Complex Systems 511, Fall 2013: Theory of Complex Systems
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Limit sets, stability, and robustness

Lecture 2, Sept. 9

A Taxonomy

deterministic dynamical systems

- iterated maps (discrete time)
  - e.g. Logistic map: $x_{t+1} = a x_t (1 - x_t)$
- differential equations (continuous time)
  - ordinary differential equations (derivatives <u>by</u> only one variable)
    - linear differential equations
    - nonlinear differential equations
  - partial differential equations (derivatives <u>by</u> more than one variable)
    - e.g. Transport eq.: $\frac{\partial u}{\partial t} = c \frac{\partial u}{\partial x}$, or Wave eq.: $\frac{\partial^2 u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Any system of ordinary differential equations can be written in the form:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \ldots, x_n) \\ \dot{x}_2 = f_2(x_1, x_2, \ldots, x_n) \\ \quad \vdots \qquad\qquad \vdots \\ \dot{x}_n = f_n(x_1, x_2, \ldots, x_n) \end{cases}$$

$$\left(\dot{x}_i = \frac{dx_i}{dt} \right)$$

(Concisely, $\dot{x} = f(x)$)

(The space of all possible $(x_1, x_2, \ldots, x_n)$ is called the "phase space".)

<u>Example</u>: Forced van der Pol oscillator

(expresses the dynamics of a particular circuit employing vacuum tubes)

$$\ddot{y} - \mu(1-y^2)\dot{y} + y - A\sin(\omega t) = 0$$

$$\underbrace{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \mu(1-x_1^2)x_2 - x_1 + A\sin x_3 \\ \dot{x}_3 = \omega \end{cases}}_{3^{rd}\text{-order nonlinear system}}$$

- System is linear if all the x_i appear to the first power only (i.e., no $x_1 x_3$, x_4^2, $\sin(x_2^4)$, e^{x_3}, ...)
- Linear ODEs are completely solvable.

$$\left(\text{Let } x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ A \text{ is an } n \times n \text{ matrix}, \ e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right.$$

Then by uniqueness of solutions to systems of linear ODEs,

$$\left. \dot{x} = Ax \Rightarrow x(t) = e^{At} x_0 \quad - \underline{\text{verify}}! \right)$$

- Nonlinear ODEs are in general not exactly solvable.
  - Some are, e.g. by separation of variables.

<u>Example</u>: $\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 \end{cases} \Rightarrow \displaystyle\int_{x_0}^{x(t)} \frac{dx}{x^2} = \int_0^t dt \Rightarrow x(t) = \frac{x_0}{1 - x_0 t}$

- Some are certainly not:

  $\dot{x} = \sin(x^2)$

- Now consider $\dot{x} = 1 - x^2$.

  We can again solve this by separation of variables:

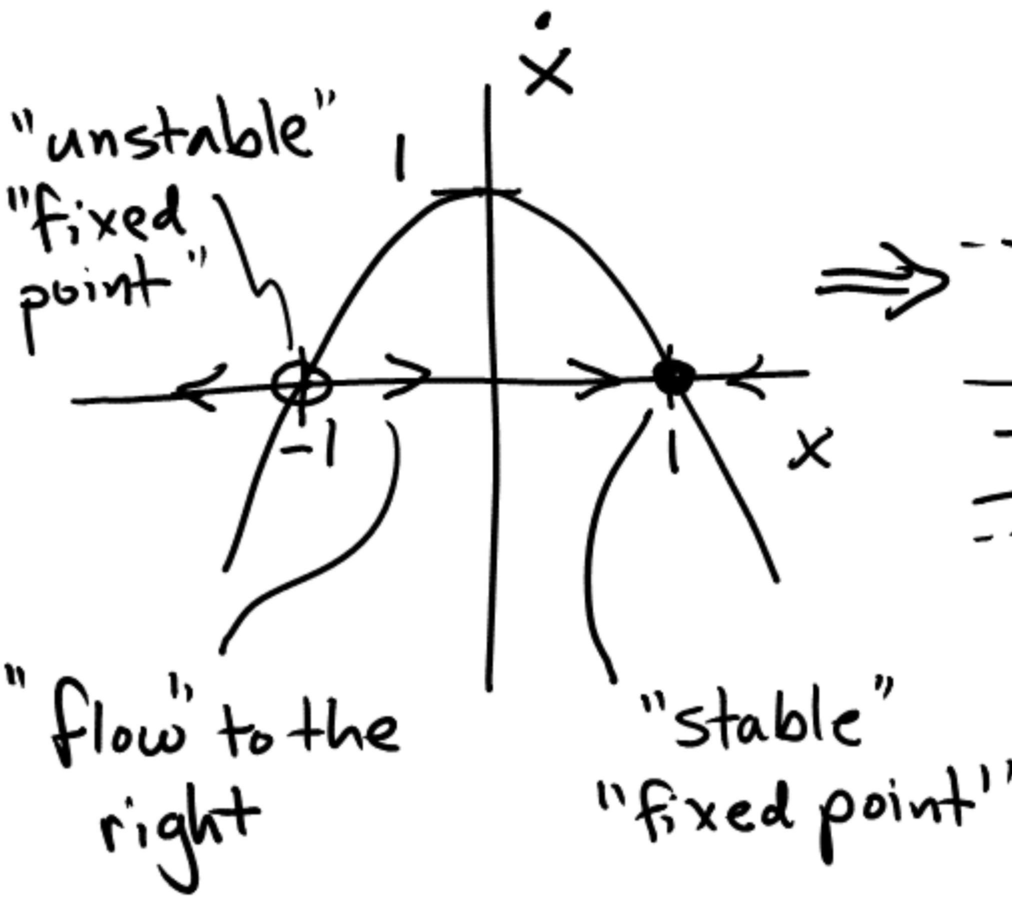
  $$\int \frac{dx}{1-x^2} = \int dt \Rightarrow \tanh^{-1}(x)\Big|_{x_0}^{x(t)} = t$$

  $$\Rightarrow x(t) = \tanh\left(t + \tanh^{-1}(x_0)\right)$$

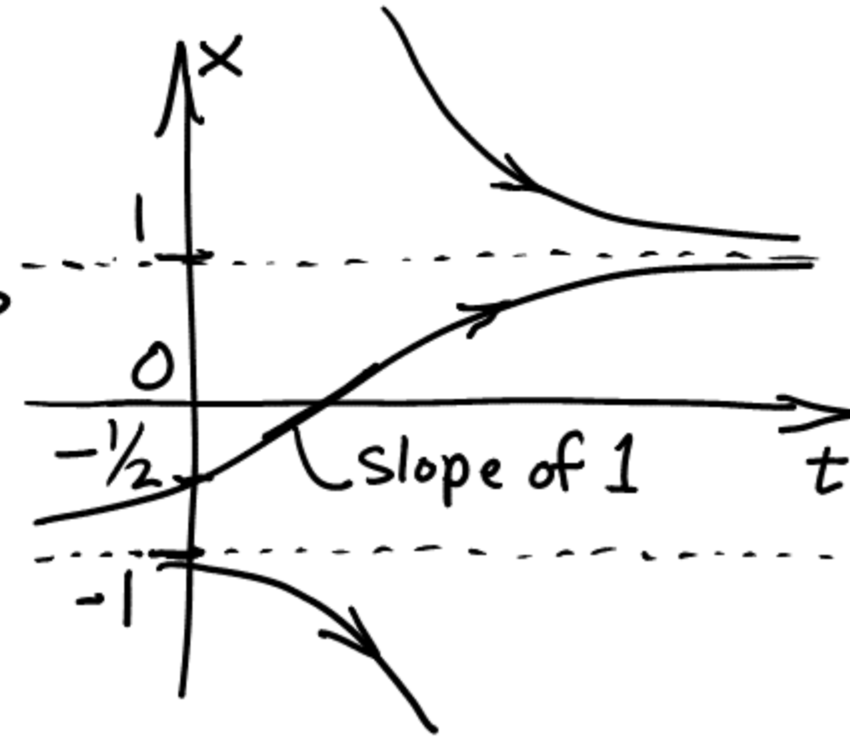
  But this is a pain to analyze. For example,

  - When $x_0 = -\frac{1}{2}$, what is the qualitative behavior for $t > 0$? For <u>arbitrary</u> x_0?

There is an easier way. Let's plot $\dot{x} = 1 - x^2$ in the $\dot{x}$ vs. x plane (called the "phase plane"):



$\Longrightarrow$



A "limit set" is a set toward which trajectories tend (formally, a set a nbd of the set approaches as $t \to \infty$, where trajectories in the nbd at $t=0$ remain there for $t \geq 0$). We just saw one type:

① fixed points

There are others:

② limit cycles

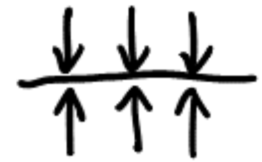
③ connected sets

④ chaotic attractors

Also as we saw, a limit set (think fixed point x^* for now) is "stable" if trajectories converge to it and "unstable" otherwise.

To be more specific,

Lyapunov stability: traj.s that start out near x^* stay near x^* for all time.



Asymptotic stability: traj.s that start out near x^* converge to x^*.

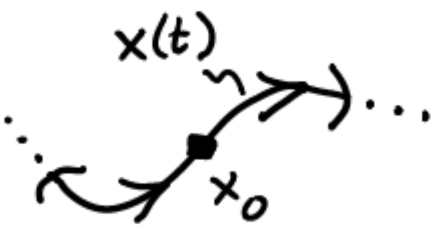
Exponential stability: traj.s that start out near x^* converge to x at $\geq$ an exponential rate of decay.

Neutral stability: Lyapunov stability but not asymptotic stability.

So far, we have taken for granted that there are traj.s that satisfy $\dot{x} = f(x)$ and that the traj. for any initial condition x_0 is unique.

<u>Existence & Uniqueness Theorem</u>: Suppose $\dot{x} = f(x)$, $x(0) = x_0$. If $f(x)$ is continuous and all $\frac{\partial f_i}{\partial x_j}$ for $i, j = 1, \dots, n$ are continuous for all x on some open connected set $D \subset \mathbb{R}^n$, then for any x_0 in D, there is exactly one traj. on some time interval $(-\tau, \tau)$ about $t = 0$ that satisfies $\dot{x} = f(x)$ and goes through x_0.

(For a proof, see Coddington & Levinson [1955], or Hirsch & Smale [1974].)



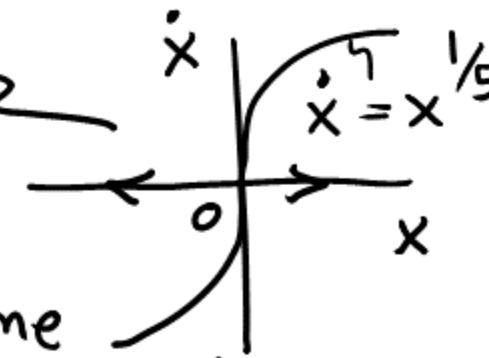
Can traj.s cross?

Note that if a partial derivative of $f(x)$ is not continuous, at x_0, then there may be more than one solution to $\dot{x} = f(x)$ at x_0:

<u>Ex</u>. Consider $\dot{x} = x^{1/5}$, $x(0)=0$. $x(t)=0$ is a solution.

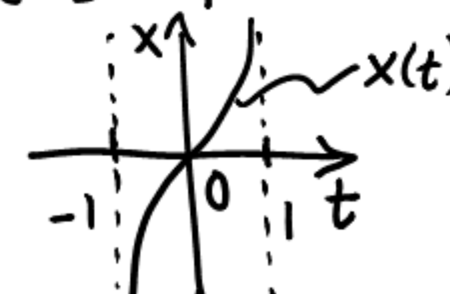
$$\int \frac{dx}{x^{1/5}} = \int dt \Rightarrow \frac{5}{4} x^{4/5} \Big|_{x(0)}^{x(t)} = t$$

So $x(t) = \left(\frac{4}{5}t\right)^{5/4}$ is also a solution. The solution is not unique because $f'(0) = \infty$.



And even when $f(x)$ meets the conditions of the theorem, $x(t)$ sometimes only exists for finite time:

<u>Ex</u>. $\dot{x} = (1+x^2)^{3/2}$, $x(0)=0 \Rightarrow \dfrac{x}{\sqrt{1+x^2}} = t$



Finally, let's take a step back and ask what does (theoretical) "robustness" mean in the context of ODEs?

<u>Meaning 1</u>: insensitivity of long-time behavior to perturbations in a given initial condition.

<u>robust</u>

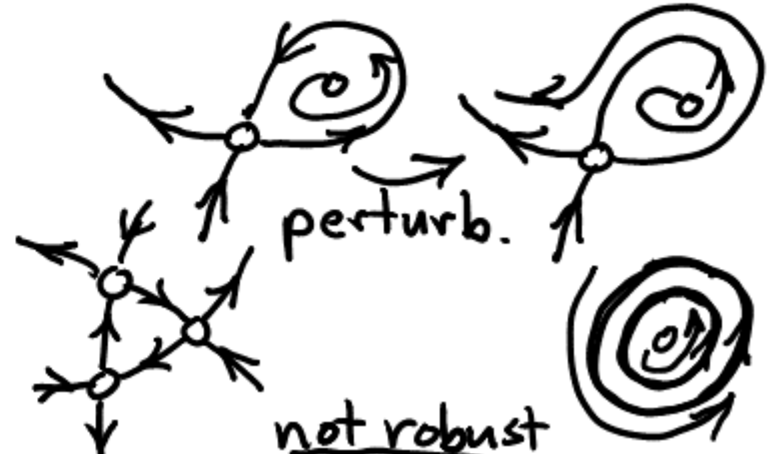
<u>not robust</u>

<u>Meaning 2</u>: insensitivity of qualitative behavior of a system $\dot{x} = f(x)$ to perturbations in $f(x)$ ("structural stability").

(originated with Andronov & Pontryagin [1937])

(See §1.7-1.9 of Guckenheimer & Holmes [1983])

<u>robust</u>



<u>not robust</u>