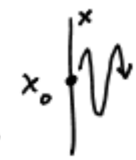
Midterm Exam - Solutions

Points possible in blue

1.a The system that produces the illustrated time series cannot have a 1-D ODE model because 1-D ODEs cannot exhibit oscillations (or even apparently oscillatory solns). This is a direct
2 consequence of the uniqueness of solns of ODEs — for the pt at right to oscillate along the shown x axis, it must feel an upward (positive) $\dot{x}$ at x_0 at one time t, but then feel a downward (negative) $\dot{x}$ at x_0 for a later time t'.



1 2-D & 3-D ODE models are possible.

1.b The notation & problem description suggest that x is a measured variable, not a parameter (a quantity that you would independently vary). Additionally, discontinuous jumps due to bifurcations appear in bifurcation diagrams,
3 $\lim_{t\to\infty} x(t)$ vs. λ for example, not in time series like $x(t)$ vs. t, where λ is presumed to be fixed. So unless a parameter in an ODE model for the system is being varied over time, the time series does not reflect a bif.

1.c This time series looks similar to that of the BZ reaction shown on slide 5 of Lecture 5, so "attractor reconstruction" would be a sensible approach: plot $y(t) = (x(t), x(t+T))$ for some delay $T>0$, then $y(t) = (x(t), x(t+T), x(t+2T))$,
3 and so on until there is a 2-D projection in which the order of traj.s is neatly maintained (see the relevant figure on slide 5). A line crossing this attractor reconstruction then gives the unimodal map characteristic of chaos.

2.a

$$\frac{du}{dt} = au + bu^3 - cu^5 \qquad \left\{ \begin{array}{l} x \overset{\text{def.}}{=} u/U \\ \tau \overset{\text{def.}}{=} t/T \end{array} \right.$$

$$\frac{U}{T}\frac{dx}{d\tau} = aUx + bU^3x^3 - cU^5x^5$$

$\vdots$

$$\frac{1}{T c U^4}\frac{dx}{d\tau} = \frac{a}{cU^4}x + \frac{b}{cU^2}x^3 - x^5$$

4 Let $U = \sqrt{b/c}\ (>0)$. Then $\frac{b}{cU^2} = \frac{b}{c}\frac{c}{b} = 1$ & $r \overset{\text{def.}}{=} \frac{a}{cU^4} = \frac{a}{c}\frac{c^2}{b^2} = \frac{ac}{b^2}$.

To solve for T, let $\frac{1}{TcU^4} = 1$. This implies:

$$T = \frac{1}{cU^4} = \frac{1}{c}\frac{c^2}{b^2} = \frac{c}{b^2}\ (>0).$$

$\boxed{2.b}$ A good place to start is by finding the fixed pts:

$0 = rx + x^3 - x^5 \Rightarrow x = 0$ OR

$$x^4 - x^2 - r = 0 \Rightarrow x^2 = \frac{1 \pm \sqrt{1+4r}}{2}$$

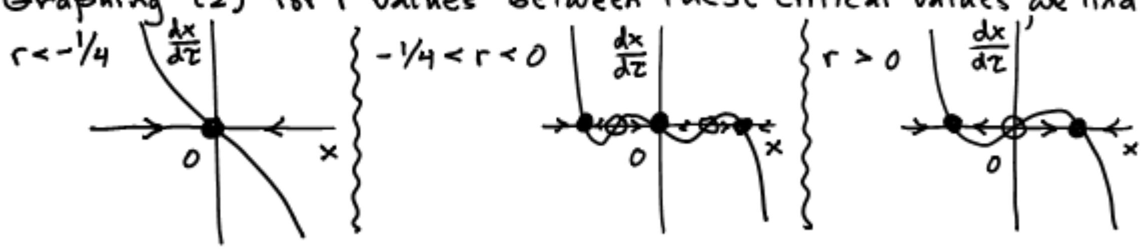
$$\Rightarrow x = \pm\sqrt{\frac{1 \pm \sqrt{1+4r}}{2}}.$$

4

Therefore, 2 saddle-node bif.s occur where $1 + 4r = 0 \Rightarrow r = -\frac{1}{4}$.

Additionally, the leading terms of (2) have the normal form of a subcritical pitchfork bif., which occurs at $r = 0$.

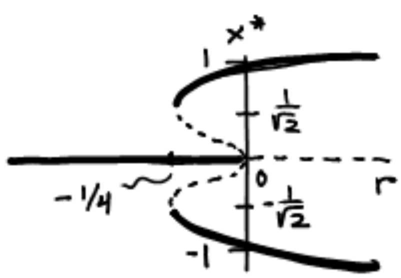
Graphing (2) for r values between these critical values we find:



These plots give us the stabilities of the fixed pts.

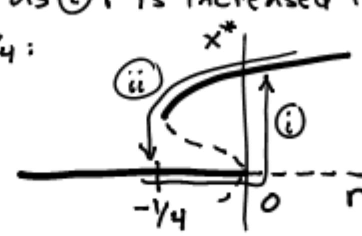
$\boxed{2.c}$ This allows us to draw the bif. diagram of x^* vs. r:

3



(We can confirm the shape by plotting $r = x^4 - x^2$ from above.)

1 Hysteresis can occur in this system, e.g. as (i) r is increased to 0 from below & then (ii) decreased to $-1/4$:

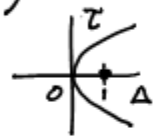


$\boxed{3}$ $\begin{cases} \dot{x} = -y + x^3 \\ \dot{y} = x + y^3 \end{cases}$

1 First, let's consider what linearization gives:

$$J(x,y) = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 & 1 \\ 1 & 3y^2 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{cases} \Delta = 1 \\ \tau = 0 \end{cases}$$



~ So linearization would suggest the origin is a center.

or a plot — A nullcline analysis is not inconsistent w/ this:

nullclines: $\begin{cases} 0 = -y + x^3 \\ 0 = x + y^3 \end{cases}$

3 However, let's (partially) convert the system to polar coordinates:

$$r^2 = x^2 + y^2 \Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y} = 2x(-y+x^3) + 2y(x+y^3)$$
$$= 2x^4 + 2y^4 \Rightarrow \dot{r} = r^3(\underbrace{\cos^4\theta + \sin^4\theta}_{\text{min: } 1/2,\ \text{max: } 1}).$$

2 So for all $r > 0$, we have $\dot{r} > 0$, & hence the origin is actually an <u>unstable spiral</u>.

$\boxed{4.a}$ $\begin{cases} \dot{x} = x(1-y) \\ \dot{y} = y(1-x) \end{cases}$

$$\begin{matrix} 0 = x(1-y) \\ 0 = y(1-x) \quad (*) \end{matrix} \Rightarrow \underline{\text{OR}} \begin{cases} x = 0 \overset{(*)}{\Rightarrow} y = 0 \\ y = 1 \overset{(*)}{\Rightarrow} x = 1 \end{cases}$$

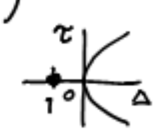
2 ~ So the system has 2 fixed pts: $(0,0)$, $(1,1)$.

1 The above also implies the system has 4 nullclines: $x = 0$, $y = 1$, $y = 0$, $x = 1$.

Linear stability analysis yields the stabilities of the fixed pts:

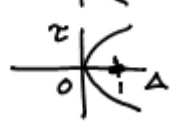
$$J(x,y) = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-y & -x \\ y & x-1 \end{pmatrix}$$

3 $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{cases} \Delta = -1 \\ \tau = 0 \end{cases} \Rightarrow$



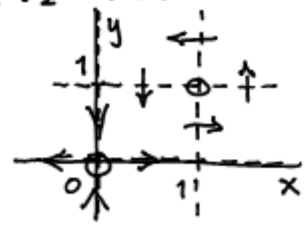
$\Rightarrow$ $(0,0)$ is a saddle pt.

$J(1,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{cases} \Delta = 1 \\ \tau = 0 \end{cases} \Rightarrow$



$\Rightarrow$ $(1,1)$ is a center.

$(0,0)$ is the only hyperbolic fixed pt & already is in diagonal
2 form. Hence it has the eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ &
corresponding eigenvectors $v_1 = \binom{1}{0}$, $v_2 = \binom{0}{1}$.
We now sketch all of the above
2 information, showing the flow
across the nullclines:

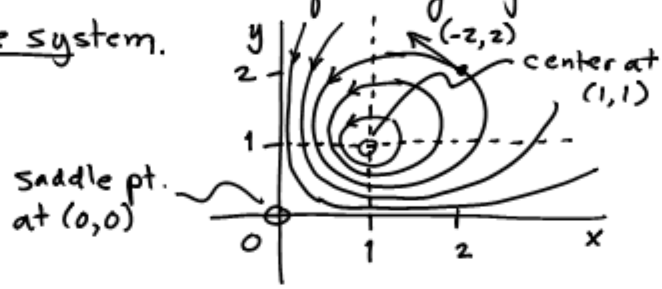


$\boxed{4.b}$ We now look for a conserved quantity, $E(x,y)$.

3
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{y(x-1)}{x(1-y)} \Rightarrow x(1-y)\,dy = y(x-1)\,dx$$
$$\Rightarrow \int (1/y - 1)\,dy = \int (1 - 1/x)\,dx$$
$$\Rightarrow \ln y - y = x - \ln x + C \Rightarrow E(x,y) = \ln(xy) - x - y = \text{constant}.$$

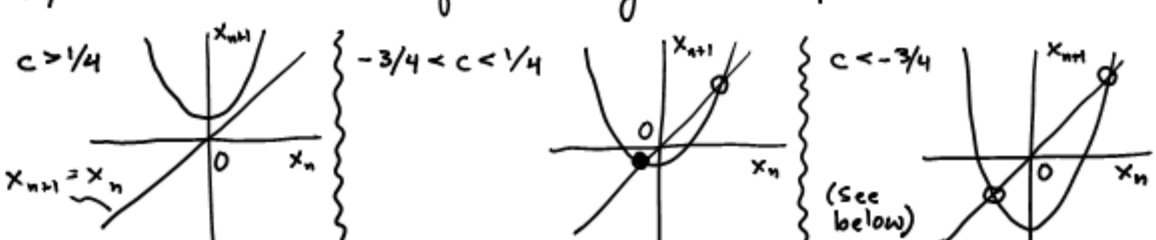
1 Thus, the traj.s of (4) must be <u>closed</u>. (Otherwise, the traj.s in the 1st quadrant would all depart from or converge to $(1,1)$, in which case we could expect all of the traj.s to have the same energy, which is not the case from the form of $E(x,y)$.)
Additionally, the presense of a conserved quantity says
1 that (4) is a <u>conservative system</u>.

1 A more complete sketch:



2

$\boxed{5.a}$ & $\boxed{5.b}$ There are 2 potential fixed pts: $x_n = x_n^2 + c \Rightarrow x_n = \dfrac{1 \pm \sqrt{1-4c}}{2}$ (*)
The form of (*) implies that the 2 fixed pts appear in a <u>tangent bif</u>. for $1 - 4c > 0 \Rightarrow c < 1/4$. This is confirmed
2 w/ sketches of the qualitatively distinct possibilities:



2 for correct stabilities

(($*$) & the above sketches indicate that at $c = 1/4$, there is just one fixed pt which is located at $x_n = 1/2$ & is half-stable.)

The slope of $x_{n+1} = x_n^2 + c$ is greater than 1 at the right fixed pt, so this fixed pt is unstable for all $c < 1/4$. The left fixed pt starts stable (the slope initially has an absolute value less

1 than 1), but then goes unstable in a <u>flip bif.</u> as the slope decreases through -1. This occurs when:

$$\frac{dx_{n+1}}{dx_n} = 2x_n = -1 \Rightarrow x_n = -1/2 \text{ lies on } x_{n+1} = x_n, \text{ which occurs}$$

1 at the c: $-1/2 = (-1/2)^2 + c \Rightarrow c = -3/4$.

Since the slope at the left fixed pt remains less than -1 for all $c < -3/4$, this fixed pt remains unstable for all $c < -3/4$.

$\boxed{5.c}$ A 2-cycle emerges when there is a p & q such that $f(p) = q$ & $f(q) = p$ & $p \neq q$ — equivalently, when there is a p (not equal to one of the fixed pts) such that $p = f(f(p)) = f^2(p)$.

$$p = f(f(p)) = f(p^2+c) = (p^2+c)^2 + c = p^4 + 2cp^2 + c^2 + c.$$

$$\Rightarrow 0 = p^4 + 2cp^2 - p + c^2 + c \quad (**)$$

The fixed pts must be solutions, so we factor $0 = p^2 - p + c$

3 from ($**$):

$$0 = (p^2 - p + c)(p^2 + \alpha p + \beta) = p^4 + (\alpha - 1)p^3 + (-\alpha + \beta + c)p^2 + (\alpha c - \beta)p + \beta c$$

To match w/ ($**$), we must have $\alpha = 1$, $-\alpha + \beta + c = 2c \Rightarrow \beta = c + 1$.

We can verify that $\alpha c - \beta = -1$ & $\beta c = c^2 + c$.

Solving $0 = p^2 + p + c + 1$ yields $p, q = \frac{-1 \pm \sqrt{1 - 4(c+1)}}{2}$. ($***$)

So a 2-cycle exists for all $1 - 4(c+1) > 0 \Rightarrow c < -3/4$ (as expected).

This 2-cycle is stable precisely when p & q are stable fixed pts of f^2 — i.e. when $-1 < \lambda < 1$, where λ is the relevant

3 multiplier.

$$\lambda = \frac{d}{dx}(f(f(x)))_{x=p} = f'(f(p))f'(p) = f'(p)f'(q) = (2p)(2q)$$

We can confirm from ($***$) that $pq = \frac{1}{4}(1 - (1 - 4(c+1))) = c + 1$.

So the 2-cycle is stable for $-1 < 4(c+1) < 1 \Rightarrow -5/4 < c < -3/4$.

$\boxed{5.d}$

2

$$y_{n+1} = r y_n (1 - y_n)$$

$$\Rightarrow a x_{n+1} + b = r(a x_n + b)(1 - (a x_n + b))$$

$$\Rightarrow x_{n+1} = \frac{r}{a}(-a^2 x_n^2 + a(1-2b)x_n + b - b^2) - \frac{b}{a}$$

$$= -ar x_n^2 + r(1-2b)x_n + \frac{rb}{a} - \frac{rb^2}{a} - \frac{b}{a}$$

$$\Rightarrow -ar = 1 \Rightarrow a = -\frac{1}{r} \; ; \; 1 - 2b = 0 \Rightarrow b = \tfrac{1}{2}$$

$$\Rightarrow x_{n+1} = x_n^2 + \frac{r}{2}\left(1 - \frac{r}{2}\right), \text{ so } c = \frac{r}{2}\left(1 - \frac{r}{2}\right).$$

1 Since we were able to find a change of variables w/ a continuous inverse, the quadratic & logistic maps are conjugate, so they exhibit equivalent dynamics. As we saw in this course, the logistic map exhibits chaos, so the quadratic map must exhibit chaos too.