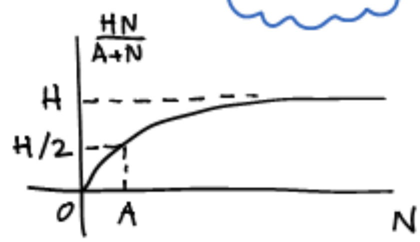
Problem Set 1 - Solutions

Points possible in blue

$\boxed{3.7.4}$



a) When $N = A$, $\frac{HN}{A+N} = \frac{H}{2}$, so A is a measure of how sensitive the intensity of the fishing or "harvesting" is to the size of the fish population, N.

2 For low A, harvesting is insensitive & proceeds at a rate H regardless of N. For high A, harvesting is sensitive to N & gradually increases w/ N, asymptotically approaching H.

b) Divide both sides by rK & top & bottom of last term by K:

$$\frac{1}{rK}\frac{dN}{dt} = \frac{N}{K}\left(1-\frac{N}{K}\right) - \frac{H}{rK}\,\frac{N/K}{A/K + N/K} \qquad (1)$$

4 Let $x = N/K$, $\tau = rt$, $a = A/K$, & $h = H/(rK)$. Then (1) becomes

$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x}$$

(In what follows, we write $\frac{dx}{d\tau}$ as $\dot{x}$.)

c) We first find all fixed pts:

3 $0 = x^*(1-x^*) - \frac{h\,x^*}{a+x^*} \Rightarrow x^* = 0$ or $1-x^* = \frac{h}{a+x^*}$

$$\Rightarrow x^{*2} + (a-1)x^* + (h-a) = 0$$

$$\Rightarrow x^* = \frac{1-a \pm \sqrt{(a-1)^2 - 4(h-a)}}{2}$$

$$= \frac{1-a \pm \sqrt{(a+1)^2 - 4h}}{2} \qquad (2)$$

3 The system always has the fixed pt $x^* = 0$. It has 2 fixed pts if there is exactly one fixed pt of (2) that is real & positive, which happens when:

$(a+1)^2 - 4h > 0 \Rightarrow h \leq (a+1)^2/4$ (i.e., both roots in (2) are real)

& either:

$a > 1$ & $a - 1 < \sqrt{(a-1)^2 - 4(h-a)} \Rightarrow 0 < -4(h-a) \Rightarrow h < a$

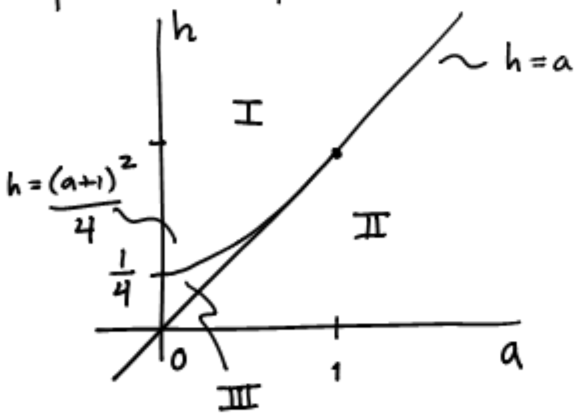
or:

$a \leq 1$ & $1 - a < \sqrt{(a-1)^2 - 4(h-a)} \Rightarrow h < a$

But $h < a \Rightarrow h \leq (a+1)^2/4$, so $h < a \Rightarrow$ exactly 2 fixed pts.

⋮ The system has 3 fixed pts if both roots of (2) are real & positive, which happens when $h \le (a+1)^2/4$ & $a<1$ & $1-a > \sqrt{(a-1)^2 - 4(h-a)} \Rightarrow 0 > -4(h-a) \Rightarrow h > a$

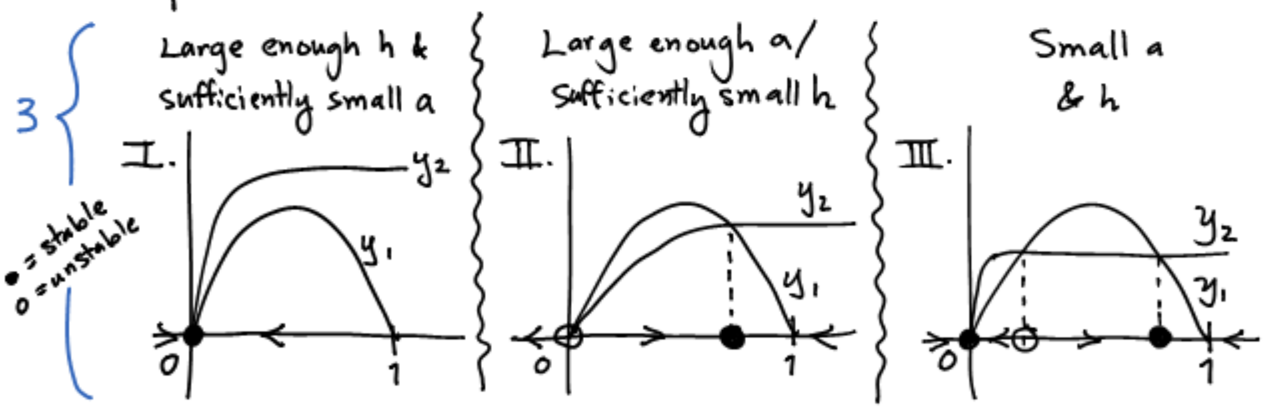
Since the system is 1-D, there can be no other limit sets, so we can draw the stability diagram of the system in (a,h) parameter space:



(We are only concerned w/ the first quadrant because the biological interpretation of a, h implies they are > 0.)

We now can simply plot the static curve $y_1 = x(1-x)$ & the curve $y_2 = \frac{hx}{a+x}$ over it. By examining the role of h & a in the plot in part (a), we can see that there are only 3 possibilities:

3



3

This geometric approach corresponds perfectly to our algebraic approach above, which is reassuring. Moreover, we can read the stabilities of the fixed pts $\dot{x} = y_1 - y_2 = 0$ off of these three plots (as shown) based on the sign of $\dot{x} = y_1 - y_2$ on either side of the fixed pt.

d) Fix a at some positive value, & look close enough at the origin that $x/a \ll 1$. Then $\dot{x}$ can be reexpressed as its Taylor expansion:

$$\dot{x} = x(1-x) - \frac{hx}{a+x} = x - x^2 - \frac{hx}{a}\left(1 - \frac{x}{a}\right)^{-1}$$

2 $$= x - x^2 - \frac{hx}{a}\left\{1 - \frac{x}{a} + \mathcal{O}\left[\left(\frac{x}{a}\right)^2\right]\right\}$$

$$= \left(1 - \frac{h}{a}\right)x - \left(1 - \frac{h}{a^2}\right)x^2 + \mathcal{O}(x^3)$$

If $1 - h/a^2 \neq 0$, i.e. $h = a^2$, then we can divide both sides by $1 - h/a^2$, and the leading terms of the dynamics are in the normal form of a transcritical bif. ($\dot{x} = rx - x^2$). So a

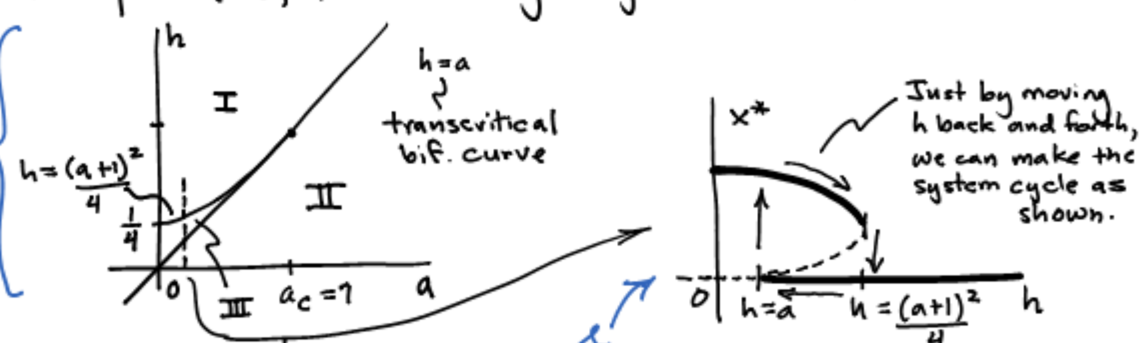
2 transcritical bif. occurs when $1 - h/a = 0$, i.e. $h = a$, w/ $x^* = 0$ unstable when $1 - h/a > 0$, i.e. $h < a$.

e) As we found in part (c), there are 2 fixed pts $\neq 0$ when $h < (a+1)^2/4$. When $h = (a+1)^2/4$, these coalesce. And when $h > (a+1)^2/4$, both nonzero fixed pts have disappeared.

2 The only 1-D bif. that fits this description is a saddle-node bif. At the bif., $x^* = (1-a)/2$. Since

2 $x^* > 0$ if biologically meaningful, we must have that $a < a_c = 1$.

f) From part (c), the stability diagram looks like:

2 

2 Hysteresis can occur if we take a path like the dotted line through the (a,h) parameter space.

6.4.2

a) $\begin{cases} \dot{x} = x(3-2x-y) \\ \dot{y} = y(2-x-y) \end{cases}$ $\quad x, y > 0$

4

Let $f(x,y) = x(3-2x-y)$, $g(x,y) = y(2-x-y)$.

$\dot{x}, \dot{y} = 0 \Rightarrow x = 0$, & $y = 0$ or $2-y=0 \Rightarrow y = 2$

or $y = 0$, & $3-2x = 0 \Rightarrow x = 3/2$

or $\begin{cases} 2x+y=3 \\ x+y=2 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=1 \end{cases}$

Hence, the fixed pts are: $(0,0), (0,2), (3/2,0), (1,1)$

The Jacobian of the system & stabilities of the fixed pts are:

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 3-4x-y & -x \\ -y & 2-x-2y \end{pmatrix}$$

4

$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \begin{cases} \Delta = 6, & \tau^2 - 4\Delta = 5^2 - 4(6) = 1 > 0 \\ \tau = 5 \end{cases}$ $\Rightarrow$ <u>unstable node</u>

$J(0,2) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \Rightarrow \begin{cases} \Delta = -2 \\ \tau = -1 \end{cases}$ $\Rightarrow$ <u>saddle point</u>

$J(3/2,0) = \begin{pmatrix} -3 & -3/2 \\ 0 & 1/2 \end{pmatrix} \Rightarrow \begin{cases} \Delta = -3/2 \\ \tau = -5/2 \end{cases}$ $\Rightarrow$ <u>saddle point</u>

$J(1,1) = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \Rightarrow \begin{cases} \Delta = 1, & \tau^2 - 4\Delta = (-3)^2 - 4(1) = 5 > 0 \\ \tau = -3 \end{cases}$ $\Rightarrow$ <u>stable node</u>

2 The nullclines are: $x=0$, $y=0$, $3-2x-y=0$, $2-x-y=0$

Altogether, the phase portrait & nullclines are shown at right. As the phase portrait indicates, the whole <u>first quadrant</u> $(x,y>0)$ is the basin of attraction for the fixed pt. $(1,1)$.

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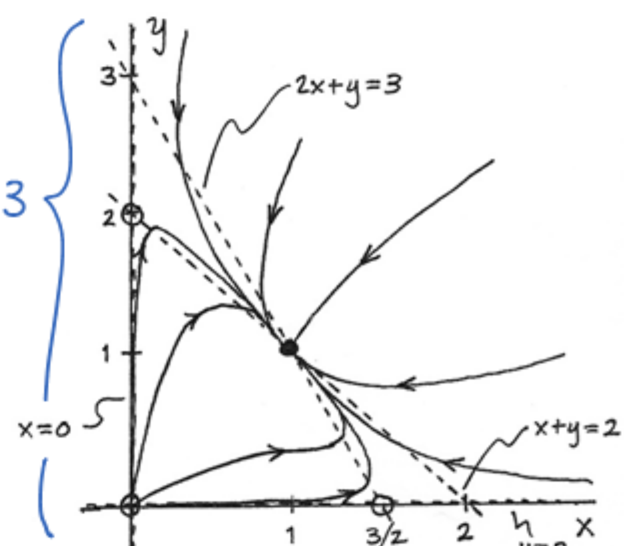
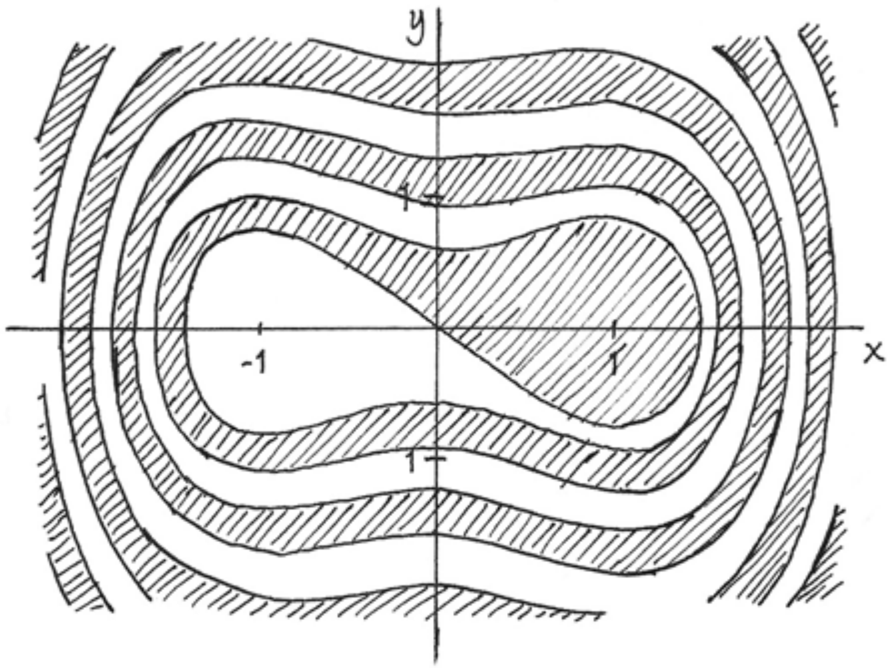


$\boxed{6.5.11}$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -by + x - x^3 \end{cases}$$

Using pplane8 in MATLAB, we find that the basin of attraction for the stable fixed pt. $(x^*, y^*) = (1,0)$ is the shaded region in the sketch below.

4



$\boxed{7.3.3}$

Choose any $\epsilon > 0$ & let $R = \{(x,y) \mid 1-\epsilon \leq x^2+y^2 \leq 2+\epsilon\}$

1 (1) The above def. indicates that R is a closed bounded subset of the plane

(2) The system $\begin{cases} \dot{x} = x - y - x^3 \\ \dot{y} = x + y - y^3 \end{cases}$

1 is composed of elementary fcns, & hence continuously differentiable everywhere on the plane.

2 (3) $\dot{x}, \dot{y} = 0 \Rightarrow \begin{cases} y = x - x^3 \\ x = y^3 - y \end{cases}$ ~ Plotting $x = (x-x^3)^3 - (x-x^3)$ implies that $(x^*, y^*) = (0,0)$ is the only fixed pt., & this is not in R.

(4) To show that the vector field points "inward" everywhere on the boundary of R, we first find the radial rate of change:

3 $2r\dot{r} = \frac{d}{dt}(r^2) = 2x\dot{x} + 2y\dot{y} = 2x(x-y-x^3) + 2y(x+y-y^3)$

$= 2x^2 + 2y^2 - 2x^4 - 2y^4 = 2r^2 - 2r^4(\cos^4\theta + \sin^4\theta)$

$\Rightarrow \dot{r} = r - r^3(\cos^4\theta + \sin^4\theta)$

Now, $\cos^4\theta + \sin^4\theta = \frac{1}{4}\cos(4\theta) + \frac{3}{4}$, so

$r - r^3 \le \dot{r} \le r - r^3/2$

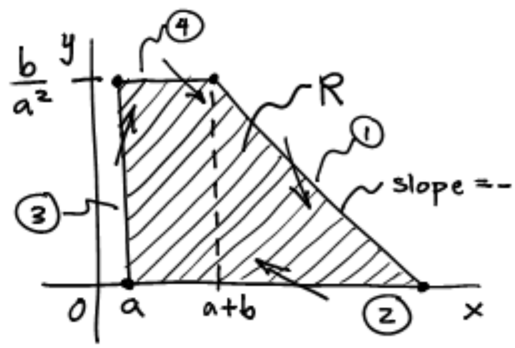
2 Hence, at $r_{min} = \sqrt{1-\epsilon}$: $\dot{r} \ge r_{min} - r_{min}^3 = \sqrt{1-\epsilon}\,\epsilon > 0$

& at $r_{max} = \sqrt{2+\epsilon}$: $\dot{r} \le r_{max} - r_{max}^3/2 = -\sqrt{2-\epsilon}\,(\epsilon/2) < 0$

2 Therefore, by the Poincaré–Bendixson Thm, the region R contains a periodic soln.

$\boxed{8.3.2}$

a) $\begin{cases} \dot{x} = a - x + x^2 y & a,b>0 \\ \dot{y} = b - x^2 y & x,y \ge 0 \end{cases}$ $(*)$



Consider the region R at right. We show this is a trapping region for $(*)$ as follows:

4 ① $\frac{\dot{y}}{\dot{x}} = \frac{b - x^2y}{a - x + x^2y} < -1 \iff b - x^2y < -(a - x + x^2y) \iff a+b < x$

② $\dot{y} = b - x^2(0) = b > 0$ for all but $y=0$, where equality holds

③ $\dot{x} = a - x + x^2y > a - x = a - a = 0$

④ $\dot{y} = b - x^2y > b - a^2(b/a^2) = 0$

for all but $x = a$, where equality holds

b) 2 $\dot{x}, \dot{y} = 0 \Rightarrow \begin{cases} 0 = a - x^* + x^{*2}y^* \\ 0 = b - x^{*2}y^* \end{cases} \underset{\text{(adding corresponding sides)}}{\Rightarrow} x^* = a+b \Rightarrow y^* = \frac{b}{(a+b)^2}$

So the system has one fixed pt: $(x^*, y^*) = \left(a+b, \frac{b}{(a+b)^2}\right)$.

3 The Jacobian of this fixed pt is:

$J(x^*, y^*) = \begin{pmatrix} -1 + 2x^*y^* & x^{*2} \\ -2x^*y^* & -x^{*2} \end{pmatrix} \Rightarrow \begin{cases} \Delta = x^{*2} = (a+b)^2 > 0 \\ \tau = -1 + b/(a+b) - (a+b)^2 \end{cases}$

⋮

The sign of τ won't change if we multiply it by $(a+b)$:

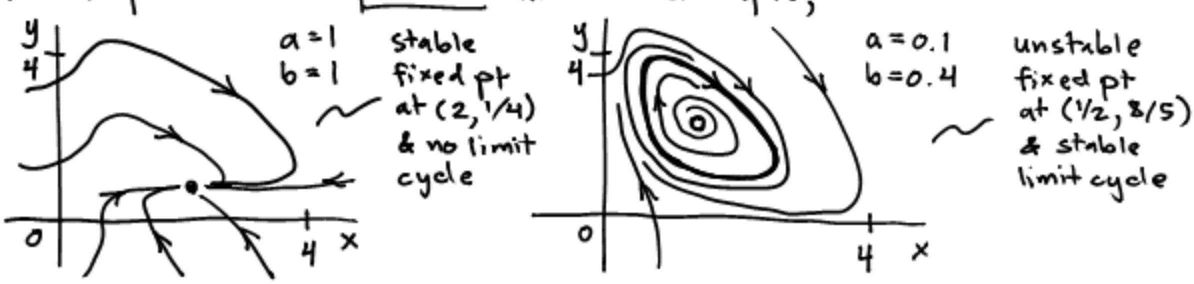
$$(a+b)\tau = -(a+b) + 2b - (a+b)^3 = (b-a) - (a+b)^3$$

Hence:

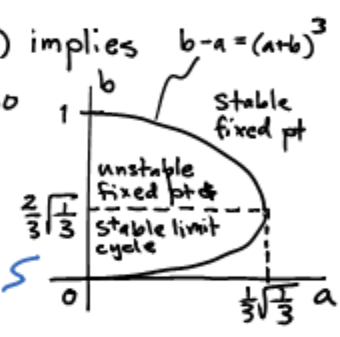
- If $b-a < (a+b)^3$, then $\tau < 0$ & (x^*, y^*) is a stable node or stable spiral.
- If $b-a > (a+b)^3$, then $\tau > 0$ & (x^*, y^*) is an unstable node or unstable spiral.

c) Eq. (5) on p. 130 gives that $\lambda_{1,2} = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$. Consider the family of lines, $a+b = c$, where c is a fixed constant. If we approach the curve $b-a = (a+b)^3$ along one of these lines,
3 then Δ stays constant (at c^2) while $\tau^2 = c^{-2}[(b-a)-(a+b)^3]$ becomes arbitrarily small. So in a sufficiently small neighborhood of the intersection w/ the curve $b-a = (a+b)^3$, the radical is purely imaginary and $\mathrm{Re}[\lambda_1] = \mathrm{Re}[\lambda_2] = \tau/2$. Hence, since $\tau/2$ crosses zero & changes sign as we move through $b-a = (a+b)^3$, the system undergoes a Hopf bifurcation across this curve.

d) By plotting the system for different parameter values using pplane8 in MATLAB, we obtain phase portraits indicating that
2 the Hopf bif. is supercritical. For example,



e) The bif. curve is given by $\tau = 0$, which from part (b) implies
2 $x^{*2} - 2x^* y^* + 1 = 0$. Part (b) also gives that $y^* = b/x^{*2}$, so $x^{*2} - 2x^*(b/x^{*2}) + 1 = 0 \Rightarrow b = \frac{1}{2}x^*(1 + x^{*2})$.
From part (c), the bif. occurs when $b-a = (a+b)^3$, so $a = b - (a+b)^3 = \frac{1}{2}x^*(1 - x^{*2}) - x^{*3} \Rightarrow a = \frac{1}{2}x^*(1 - x^{*2})$.



25