Problem Set 2 – Solutions

(I give the approximate level of detail I expect in these solns. For another example of an ideal problem set write-up, see Problem Set 1 - Solutions.)

9.2.1

a) The Jacobian of the Lorenz system is

$$J(x,y,z) = \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

As given on p. 314 of the course text, C^+ & C^- are located at $(x^*,y^*,z^*) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$, so the Jacobian at C^+, C^- is:

$$J(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1) = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \pm\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{pmatrix}$$

So the characterist eq. for C^+, C^- is

$\det[J(x^*,y^*,z^*) - \lambda I]$

simply expanding & then grouping by powers of λ.

$$= (-\sigma-\lambda)[(-1-\lambda)(-b-\lambda) + b(r-1)] + \sigma[-b(r-1) + b + \lambda] = 0$$

$$\Rightarrow \lambda^3 + (\sigma+b+1)\lambda^2 + (r+\sigma)b\lambda + 2b\sigma(r-1) = 0 \qquad (1)$$

b) $(\lambda - i\omega)(\lambda + i\omega)(\lambda - \lambda_3) = 0 \Rightarrow \lambda^3 - \lambda_3\lambda^2 + \omega^2\lambda - \lambda_3\omega^2 = 0$

So if $\lambda_1, \lambda_2 = \pm i\omega$, then from (1),

$$\begin{cases} \lambda_3 = -(\sigma+b+1) & (2) \\ \omega^2 = (r+\sigma)b & (3) \\ -\lambda_3\omega^2 = 2b\sigma(r-1) & (4) \end{cases}$$

Substituting (2) & (3) into (4) gives $(\sigma+b+1)(r+\sigma)b = 2b\sigma(r-1)$.

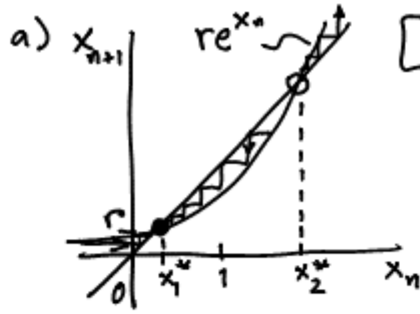
Rearranging gives $r = r_H = \sigma\left(\dfrac{\sigma+b+3}{\sigma-b-1}\right)$.

We must assume $\sigma > b+1$ since otherwise r_H is not positive & finite (no Hopf bif. at C^+, C^- for $r>0$).

c) As found in part (b), $\lambda_3 = \boxed{-(\sigma+b+1).}$

10.4.1

a)



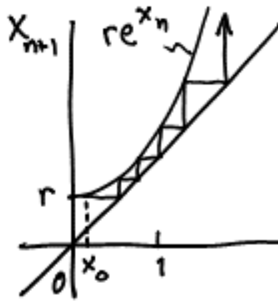
$\boxed{r<1/e}$

As $n\to\infty$:

$x_n \to x_1^*$ if $x_0 < x_2^*$

$x_n \to x_2^*$ if $x_0 = x_2^*$

$x_n \to \infty$ if $x_0 > x_2^*$



$\boxed{r>1/e}$

As $n\to\infty$:

$x_n \to \infty$ for all x_0

&

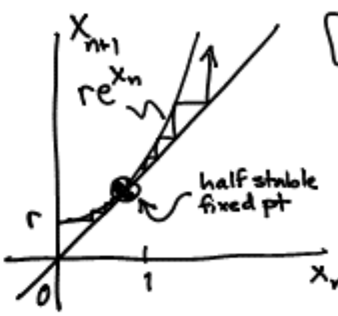
b) At the pt of a tangent bif., $x = re^x$ (5) has one soln, &

$\frac{d}{dx}(x) = \frac{d}{dx}(re^x)$ (6) at this pt.

(6) $\Rightarrow 1 = re^x$ (7) $\overset{(5)}{\Rightarrow} x = 1$ (8)

(7) & (8) $\Rightarrow 1 = re \Rightarrow r = 1/e$.

The tangent bif. is the discrete-time version of the saddle-node bif. of 1-D ODEs.



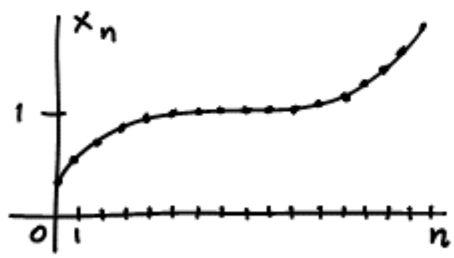
$\boxed{r=1/e}$

As $n\to\infty$:

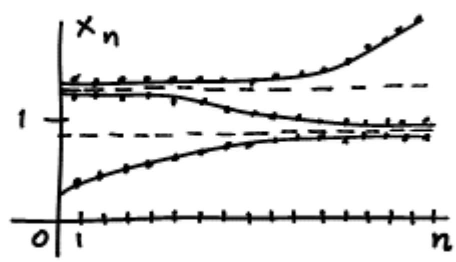
$x_n \to 1$ if $x_0 \le 1$

$x_n \to \infty$ if $x_0 > 1$

c)



$\boxed{\text{r just above } 1/e}$

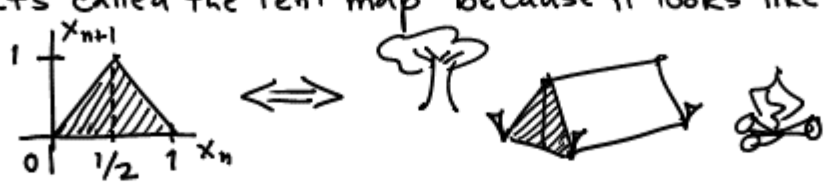


$\boxed{\text{r just below } 1/e}$

(Solid lines interpolating the iterated pts are simply to guide the eye.)

9.4.2

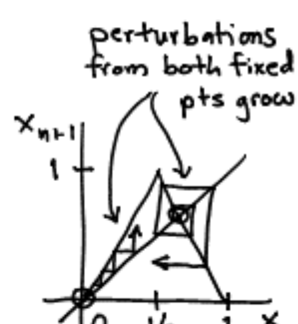
a) It's called the "tent map" because it looks like the end of a tent:



b) Fixed pts:

$x^* = 2x^* \Rightarrow x^* = 0$ (on $0 \le x_n \le 1/2$, so valid)

$x^* = 2 - 2x^* \Rightarrow x^* = 2/3$ (on $1/2 \le x_n \le 1$, so valid)



Stability of all fixed pts <u>& orbits</u>:

Let η_n be a small deviation about a fixed pt or pt on an orbit.

Let η_{n+p} be this same pt after p iterations of the tent map.

By the process leading up to Eq. 2 on p. 330:

$$\eta_{n+p} = \left[\prod_{k=0}^{p-1} f'(x_{n+k})\right]\eta_n$$ (if $p>0$, & p & η_n small enough that x_n & $x_n+\eta_n$ are alway on the same side of $1/2$). (*)

Observe: $f'(x_{n+k}) = 2$ for all x_{n+k} on $(0, 1/2)$.

$f'(x_{n+k}) = -2$ for all x_{n+k} on $(1/2, 1)$

So $|\eta_{n+p}| = 2^p|\eta_n| \Rightarrow |\eta_{n+p}| > |\eta_n|$ (assuming (*)).

Thus small deviations from all fixed pts & orbits grow, so <u>all fixed pts & orbits are unstable</u>. (Note: Although $x^* = 0$ is unstable, the countably infinite # of sequences $x_0, x_1, x_2, \ldots$ that contain 1 — proceeded in general by $1/2$, and so on — all end up at 0.)

c) Let $x_{n+1} = f(x_n)$ denote the tent map. Then then a period-2 orbit (2-cycle) exists if & only if there are 2 pts x_1, x_2 such that $x_1 = f(x_2)$ & $x_2 = f(x_1)$ but $x_1 \neq x_2$, i.e. $x_1 = f(f(x_1))$ but $x_1 \neq f(x_1)$.

There are 3 possibilities:

i) The 2-cycle is entirely on $L = [0, 1/2]$:

$x = f(f(x)) = 2(2x) = 4x \Rightarrow x = 0$ — one of the fixed pts found above—<u>not</u> a 2-cycle

ii) The 2-cycle is entirely on $R = [1/2, 1]$:

$x = f(f(x)) = 2 - 2(2-2x) = -2 + 4x \Rightarrow x = 2/3$

iii) The 2-cycle has one pt on L & one pt on R:

$x = 2 - 2(2x) = 2 - 4x \Rightarrow x_1 = \boxed{2/5} \Rightarrow x_2 = 2(2/5) = \boxed{4/5}$

This 2-cycle is <u>unstable</u> by the analysis in part (b).

d) For a period-3 orbit (3-cycle), there are 4 possibilities:

i) $x = 2(2(2x)) = 8x \Rightarrow x = 0$

ii) $x = 2 - 2(2(2x)) = 2 - 8x \Rightarrow x_1 = \boxed{2/9}, x_2 = \boxed{4/9}, x_3 = \boxed{8/9}$

iii) $x = 2 - 2(2 - 2(2x)) = -2 + 8x \Rightarrow x_1 = \boxed{2/7}, x_2 = \boxed{4/7}, x_3 = \boxed{6/7}$

iv) $x = 2 - 2(2 - 2(2 - 2x)) = 6 - 8x \Rightarrow x = 2/3$

For a period-4 orbit (4-cycle), there are 6 possibilities:

i) $x = 2(2(2(2x))) = 16x \Rightarrow x = 0$

ii) $x = 2 - 2(2(2(2x))) = 2 - 16x \Rightarrow x_1 = \boxed{2/17}, x_2 = \boxed{4/17}, x_3 = \boxed{8/17}, x_4 = \boxed{16/17}$

iii) $x = 2 - 2(2 - 2(2(2x))) = -2 + 16x \Rightarrow x_1 = \boxed{2/15}, x_2 = \boxed{4/15}, x_3 = \boxed{8/15}, x_4 = \boxed{14/15}$

iv) $x = 2 - 2(2(2 - 2(2x))) = -6 + 16x \Rightarrow x = 2/5$

v) $x = 2-2(2-2(2-2(2x))) = 6-16x \Rightarrow x_1 = \boxed{6/17}, x_2 = \boxed{12/17}, x_3 = \boxed{10/17}, x_4 = \boxed{14/17}$

vi) $x = 2-2(2-2(2-2(2-2x))) = -10+16x \Rightarrow x = 2/3$

All of the periodic orbits above are unstable by the analysis in part (b).

11.3.1

a) Let's call the middle-halves Cantor Set C_{MH}. Each step in the construction of C_{MH} (after the first) consists of $m=2$ copies of the previous step, each scaled down by a factor of $r=4$. Thus, the similarity dimension of C_{MH} is

$$d = \frac{\ln m}{\ln r} = \frac{\ln 2}{\ln 4} = \boxed{\frac{1}{2}}.$$

b) Let L_n denote the summed lengths of the line segments in the n^{th} step of the construction of C_{MH}. By the definition of C_{MH}, $L_n = (1/2)^n$, & $\lim_{n\to\infty} L_n = \lim_{n\to\infty} (1/2)^n = 0$. So C_{MH} has a measure of 0.

11.4.6

a) The set of initial conditions (I.C.s) that escape after 1 iteration is found as follows:

$$\left.\begin{aligned} 1 < rx_0 &\Rightarrow 1/r < x_0 \\ 1 < r(1-x_0) &\Rightarrow x_0 < 1-1/r \end{aligned}\right\} \Rightarrow 1/r < x_0 < 1-1/r.$$

The set of I.C.s that escape after exactly 2 iterations is found as follows:

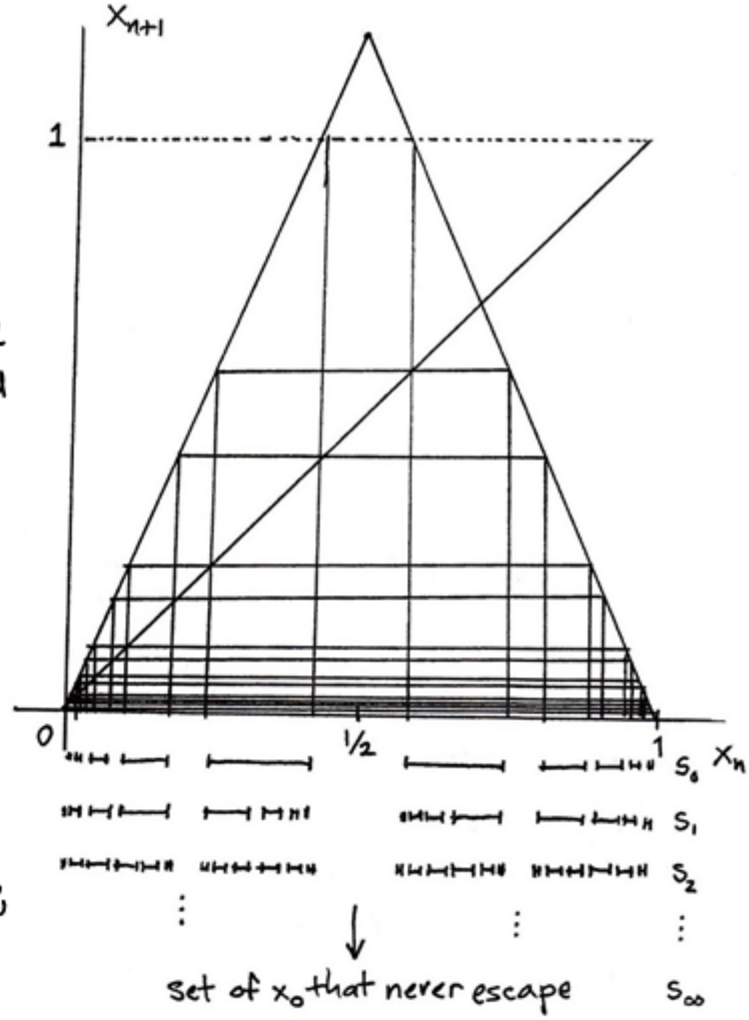
$$\underbrace{1/r < f(x_0) < 1-1/r}_{\text{(tent map)}} \Rightarrow \begin{cases} 1/r < rx_0 < 1-1/r \Rightarrow 1/r^2 < x_0 < 1/r - 1/r^2 \\ 1/r < r(1-x_0) < 1-1/r \Rightarrow 1 - 1/r + 1/r^2 < x_0 < 1 - 1/r^2 \end{cases}$$

Altogether, the set of initial conditions that escape after 1 or 2 iterations is:

$$\boxed{(1/r^2, 1/r - 1/r^2) \cup (1/r, 1-1/r) \cup (1 - 1/r + 1/r^2, 1 - 1/r^2)}$$

b) The set of initial conditions x_0 that never escape is the fractal S_∞ illustrated below (on the next page). Based on similarities in construction w/ the Cantor Set, we propose (w/out proof) that S_∞ is a topological Cantor Set — "totally disconnected," but w/ no "isolated pts."

Let L_n denote the summed lengths of the line segments in the n^{th} step, S_n, of the construction of S_∞ indicated w/ the illustration at right. Then step S_{n+1} contains $m=2$ copies of step S_n, each scaled down by a factor r (which can be inferred simply by iterating $rx_0 \le 1 \Rightarrow x_0 \le 1/r$, $r(rx_0) \le 1 \Rightarrow x_0 \le 1/r^2$, etc. & from the symmetry of the problem). This implies that L_n can be bounded by $0 < L_n < (2/r)^{n+1}$. Since $r>2$, we have $\lim_{n\to\infty} (2/r)^{n+1} = 0$, so (by the squeeze thm for sequences) S_∞ has a measure of 0.



c) From the observation in part (b), the similarity dimension of S_∞, the set of x_0 that never escapes, is

$$d = \frac{\ln m}{\ln r} = \boxed{\frac{\ln 2}{\ln r}}.$$

Note that $0 < d < 1$ since the problem assumed $r > 2$.

d) First, $f'(x) = r$ for all $x \in (0, 1/2)$ & $f'(r) = -r$ for all $x \in (1/2, 1)$. So, by the def. of the Lyapunov exponent on p. 367,

$$\lambda = \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \right\} = \lim_{n\to\infty} \ln r = \boxed{\ln r} > 0 \quad \text{for } r > 1.$$

Since the problem assumes $r > 2$, we have that $\lambda > 0$ at each pt in S_∞ except 0, 1.