# Problem Set 3 - Solutions

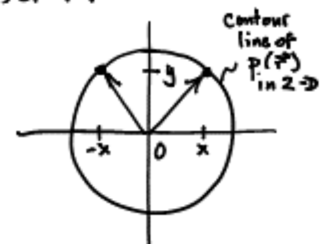
(In the below, I just show the expected math. In your work you should give justification, often from the notes, for each step.)

$\boxed{1.}$

a) Write $\vec{r}$ as $(x,y,z)$, & $\nabla P_t(\vec{R})$ as $(a,b,c)$ (as $\vec{r}$-indep. vector).

$$\int \rho(\vec{r})(\vec{r}\cdot\nabla)P_t(\vec{R})\,d^3\vec{r}$$
$$= \int \rho(\vec{r})[(x,y,z)\cdot(a,b,c)]\,d^3\vec{r}$$
$$= a\int x\rho(\vec{r})\,d^3\vec{r} + b\int y\rho(\vec{r})\,d^3\vec{r} + c\int z\rho(\vec{r})\,d^3\vec{r}. \quad (1)$$

Observe $\rho((-x,y,z)) = \rho((x,y,z))$. So for $y$ & $z$ fixed $x$ is odd & $\rho((x,y,z))$ is even, so the integral over $x$ of $x\rho((x,y,z))$ is zero. Hence




$$\int x\rho(\vec{r})\,d^3\vec{r} = 0$$

By analogous reasoning, all 3 integrals in (1) are zero, so (1) is zero.

b) $\int \rho(\vec{r})(\vec{r}\cdot\nabla)^2 P_t(\vec{R})\,d^3\vec{r}$

$$= \int \rho(\vec{r})\left[(x,y,z)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\right]^2 P_t(\vec{R})\,d^3\vec{r}$$
$$= \int \rho(\vec{r})\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)^2 P_t(\vec{R})\,d^3\vec{r} \quad (2)$$
$$= \int \rho(\vec{r})\left(x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + z^2\frac{\partial^2}{\partial z^2} + 2xy\frac{\partial^2}{\partial x\partial y} + 2xz\frac{\partial^2}{\partial x\partial z} + 2yz\frac{\partial^2}{\partial y\partial z}\right) P_t(\vec{R})\,d^3\vec{r}$$

By reasoning analogous to that above, all the cross terms such as $\int xy\,\rho(\vec{r})\,d^3\vec{r}$ are zero. So (2) reduces to

$$\int x^2\rho(\vec{r})\,d^3\vec{r}\,\frac{\partial^2}{\partial x^2}P_t(\vec{R}) + \int y^2\rho(\vec{r})\,d^3\vec{r}\,\frac{\partial^2}{\partial y^2}P_t(\vec{R}) + \int z^2\rho(\vec{r})\,d^3\vec{r}\,\frac{\partial^2}{\partial z^2}P_t(\vec{R})$$
$$= \langle x^2\rangle \nabla^2 P_t(\vec{R}) \quad \text{(by the isotropy of } \rho(\vec{r}),\ \langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle)$$
$$= \frac{3\langle x^2\rangle}{3}\nabla^2 P_t(\vec{R})$$
$$= \frac{\langle \vec{r}^{\,2}\rangle}{3}\nabla^2 P_t(\vec{R})$$

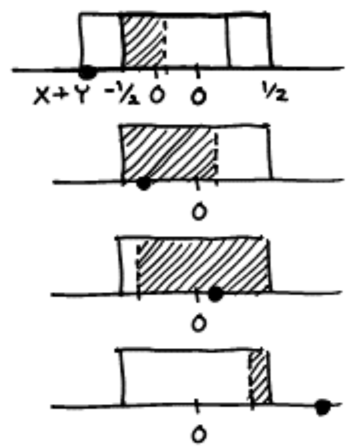
$\boxed{2.}$

a) $E[e^{itX}] = \frac{1}{2}e^{it(1)} + \frac{1}{2}e^{it(-1)} = \cos t$

b) $E[e^{itX}] = \int_{-c}^{c} \frac{1}{2c} e^{itx}\,dx = \int_{-c}^{c} \frac{1}{2c}\cos(tx)\,dx + i\overbrace{\int_{-c}^{c}\frac{1}{2c}\sin(tx)\,dx}^{0,\text{ by odd symmetry}}$

$$= \frac{1}{2c}\,\frac{\sin(tx)}{t}\Big|_{-c}^{c} = \frac{\sin(ct)-\sin(-ct)}{2ct} = \frac{\sin(ct)}{ct}$$

c) Observe that the sum of 2 i.i.d. r.v.s, $X+Y$, each w/ uniform dist. on $(-\frac{1}{2},\frac{1}{2})$, is a r.v. w/ a triangular distribution $1-|x|$, where $x\in(-1,1)$. We see this as follows (see below): We move the final value of $X+Y$ from left to right & integrate over the shaded region, which corresponds to the interval of values of $X$ that can yield the specified value of $X+Y$. Computed pointwise, the value of this integral is clearly $1-|x|$, $x\in(-1,1)$.




Hence, the ch.f. of the triangular dist. is:

$$E[e^{it(X+Y)}] = E[e^{itX}]E[e^{itY}]$$
$$= \left(\frac{\sin(t/2)}{t/2}\right)^2$$
$$= \frac{2(1-\cos t)}{t^2} \quad (\text{Since } \cos^2\theta = 1-2\sin^2\theta)$$

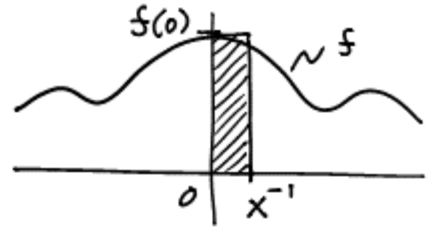
3.

a) The Cauchy distribution is a stable law w/ $\alpha=1$ & $\kappa=0$. So the ch.f. of $X_k$ is $E[e^{itX_k}] = \exp(itc - b|t|)$ where $c, b$ are constants. Therefore, the ch.f. of $\frac{1}{n}(X_1+X_2+\cdots+X_n)$ is

$$E[e^{i(t/n)(X_1+X_2+\cdots+X_n)}]$$
$$= \prod_{k=1}^{n} E[e^{i(t/n)X_k}] =$$
$$= \prod_{k=1}^{n} \exp(i(t/n)c - b|t/n|)$$
$$= \exp\left(i\sum_{k=1}^{n}(t/n)c - b\sum_{k=1}^{n}|t/n|\right)$$
$$= \exp(itc - b|t|) = E[e^{itX_1}].$$

Since $\frac{1}{n}(X_1+X_n+\cdots+X_n)$ & $X_1$ have the same ch.f. they have the same distribution.

b) Take $x$ to be positive.




$\Pr[1/X_k > x] = \Pr[X_k < x^{-1}]$

As we take $x \to \infty$, the rectangular area at right becomes an increasingly better approximation of $\Pr[X_k < x^{-1}]$, i.e. $\Pr[1/X_k > x] \sim f(0)/x$ as $x \to \infty$.

By the same argument reflected across 0,

$$\Pr[1/X_k < -x] \sim f(0)/x \quad \text{as } x \to \infty.$$

Hence,

(i.) $$\lim_{x\to\infty} \frac{\Pr[1/X_k > x]}{\Pr[|1/X_k| > x]} = \lim_{x\to\infty} \frac{\Pr[1/X_k > x]}{\Pr[1/X_k > x] + \Pr[1/X_k < -x]}$$

$$= \lim_{x\to\infty} \frac{f(0)/x}{f(0)/x + f(0)/x} = \frac{1}{2} \in [0,1]$$

(ii.) $$\Pr[|1/X_k| > x] = \Pr[1/X_k > x] + \Pr[1/X_k < -x] \sim x^{-1} f(0) + x^{-1} f(0)$$

$$= x^{-1}\, 2f(0)$$

$2f(0)$ is trivially slowly varying $\left(\lim_{x\to\infty} \frac{2f(0)}{2f(0)} = 1\right)$.

The above show that the mean $\frac{1}{n}\left(\frac{1}{X_1} + \frac{1}{X_2} + \cdots + \frac{1}{X_n}\right)$ satisfies the condition of the thm on slides 14-15 of Lecture 6, w/ $\theta = \frac{1}{2}$ & $\alpha = 1$. So $K = 2\theta - 1 = 0$, & by case b on slide 15, the mean is a Cauchy r.v.

$\boxed{4.}$ (**) is $U_k(x) = p x U_{k+1}(x) + q x U_{k-1}(x) \qquad 1 < k < N$

For simplicity, consider just one $x$.

1 First, by substitution of the form,

$$U_k(x) = A(x)\lambda_+^k(x) + B(x)\lambda_-^k(x) \qquad (\ddagger)$$

into (**), & the def. of $\lambda_\pm^k(x)$, this form does satisfy (**), so the $U_k(x)$ do provide a soln to (**).

2 Second, for arbitrary initial values $c_0$ & $c_1$, we can set $A(x)$ & $B(x)$ in $(\ddagger)$ such that $U_0(x) = c_0$ & $U_1(x) = c_1$, since this is just a 2-D linear problem.

③ Third, let $V_k(x)$ be an arbitrary soln satisfying (**), & choose $c_0$ & $c_1$ to be $V_0(x)$ & $V_1(x)$, respectively. Then by (**), $V_2(x)$ must be just the same as the $U_2(x)$ implied by (‡) w/ $A(x)$ & $B(x)$ as found in part ②. Likewise, $V_3(x)$ must be the same as $U_3(x)$, & so on. So $V_k(x) = U_k(x)$ for all $k$, i.e. all $V_k(x)$ must have the form (‡).

**5.** First, we evaluate $\lambda_\pm(1)$ 2 different ways:

$$\lambda_\pm(1) = \frac{1 \pm \sqrt{1-4pq}}{2p} = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm \sqrt{(1-2p)^2}}{2p} = \frac{1 \pm |1-2p|}{2p}$$

$$= \frac{1 \pm \sqrt{1-4(1-q)q}}{2p} = \frac{1 \pm \sqrt{(1-2q)^2}}{2p} = \frac{1 \pm |1-2q|}{2p}$$

Suppose $p = \frac{1}{4}$. Then the above indicates $\lambda_+(1) = 2(1+\frac{1}{2}) = 3$
& $\lambda_-(1) = 1$

By the def. of $U_k$, $U_k(x) \leq U_k(1) \leq 1$, where the 2nd inequality holds since $U_k(1)$ is the prob. of going bankrupt starting from k. For $p = \frac{1}{4}$,

$$U_k(x) = A(x)3^k + B(x).$$

So for any $A(x) \neq 0$, there is some $k$ large enough that

$$U_k(x) \notin [0,1],$$

which is not allowed for a probability. So, $A(x) = 0$ for all $x$. Hence,

$$U_k(x) = B(x)\lambda_-^k(x).$$

But the initial condition $U_0(x) = 1$ still holds, so $B(x) = 1$ &

$$U_k(x) = \lambda_-^k(x).$$

The above evaluation of $\lambda_\pm(1)$ gives that if $p \leq q$, then

$$\lambda_-^k(1) = \frac{1-(1-2p)}{2p} = 1,$$

and if $p \geq q$, then

$$\lambda_-^k(1) = \frac{1-(1-2q)}{2p} = \frac{q}{p}.$$

So, $U_k(1) = \begin{cases} 1 & p \leq q \\ (q/p)^k & p < q \end{cases}.$