Problem Set 3

Due: Beginning of class (at 2 p.m.) on Monday, Oct. 21.

1) In class, we derived the diffusion equation from a random walk in arbitrary dimensions. Here we verify the two unjustified steps in that derivation, completing the demonstration. For simplicity, consider the case of just three dimensions ($d = 3$). Verify that:

(i) $\int p(\mathbf{r})\,(\mathbf{r}\cdot\nabla)P_t(\mathbf{R})\,d^3\mathbf{r} = 0$, and

(ii) $\int p(\mathbf{r})\,(\mathbf{r}\cdot\nabla)^2 P_t(\mathbf{R})\,d^3\mathbf{r} = \dfrac{\langle \mathbf{r}^2\rangle}{6}\nabla^2 P_t(\mathbf{R})$.

2) Compute the characteristic functions of the following three random variables, expressing each answer in terms of powers of t and trigonometric functions to the first power only.

(i) A coin flip between the values -1 and 1: $\Pr[X = 1] = Pr[X = -1] = 1/2$.

(ii) A random variable with a uniform distribution on $(-c, c)$.

(iii) A random variable with a triangular distribution: $1 - |x|$, where $x \in (-1, 1)$.
(Hint: For part (iii), consider using the result of part (ii).)

3) Stable laws.

(i) Show that the average of n identical Cauchy random variables $X_1, X_2, \ldots$ is just X_1. (Note what this says: Suppose we want to improve our estimate of a Cauchy distributed measurement. So we take n such measurements and average them together. Remarkably, this average is no more accurate than our first measurement!)

(ii) Now suppose that we have a sequence of independent and identically distributed (i.i.d.) variables $X_1, X_2, \ldots,$ and assume that each of these X_k has a density that is symmetric about 0 and continuous and positive at 0. Show that

$$\frac{1}{n}\left(\frac{1}{X_1} + \cdots + \frac{1}{X_n}\right) \tag{1}$$

converges to a Cauchy random variable.
(Hints: First show that $\Pr[1/X_k > x]$ and $\Pr[1/X_k < -x]$ each go like $f(0)/x$ in the limit $x \to \infty$. Using the theorem in the class notes that introduces θ and α, show that this implies $\theta = 1/2$ and $\alpha = 1$. Use the general form of the characteristic function of a stable law to conclude that (1) must approach a Cauchy random variable.)

4) Give an informal proof of the claim in the class notes that: "Every soln [of (**)] must therefore [in general] be of the form: $U_k(x) = A(x)\lambda_+^k(x) + B(x)\lambda_-^k(x)$, w/ $A(x)$, $B(x)$ arbitrary." (Express as a carefully reasoned paragraph.)

5) Suppose that you are gambling according to the classical ruin problem that we described in class, but this time against an infinitely rich adversary. Show that your probability of ruin is 1 if $p \le q$ and $(q/p)^k$ otherwise.
(Hints: First give an incontrovertible argument for why $A(x)$ must be zero. Then find the new form for $U_k(x)$ that this implies. Finally, use the fact that, by definition, $U_k(1)$ is the probability of going bankrupt in finite time to rearrange $\lambda_-^k(x)$ and extract the result.)