Problem Set 5 – Solutions

1.

a) From the problem description, a suitable eq. expressing scale invariance of the pattern of open & closed sites at the phase transition is:

$$\underbrace{\rho_c^s}_{\substack{\text{Prob. that site}\\ \text{of black lattice}\\ \text{is open at}\\ \text{phase transition.}}} = f(\rho_c^s) = \underbrace{{\rho_c^s}^3 + 3{\rho_c^s}^2(1-\rho_c^s)}_{\substack{\text{Prob. that site}\\ \text{of red lattice}\\ \text{is open at}\\ \text{phase transition.}}} \qquad (*)$$

(Equated because black & red lattice have the same graph structure.)

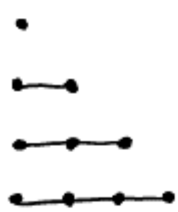
b) $(*)$ reduces to $0 = 2{\rho_c^s}^3 - 3{\rho_c^s}^2 + \rho_c^s = \rho_c^s(\rho_c^s - 1)(2\rho_c^s - 1)$

$\Rightarrow \rho_c^s = 0, 1/2, 1.$

We can expect extremal values (0 & 1) to satisfy our condition $(*)$ for scale invariance even though phase transitions do not occur at these values. $\rho_c^s = 1/2$ is the only nonextremal root of $(*)$, suggesting that it is the value (ρ_c^s) at which a phase transition occurs.

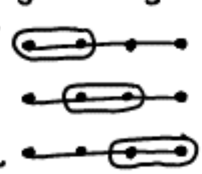
2.

a) Computing directly by inspection:



n	graph	I_n
0		0 (chosen to be 0 so the rest of the math is as elegant as possible.)
1		1
2		0
3		$\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}$
4		$\frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 = \frac{1}{3}$

equal prob. of 1/3 for each initial configuration

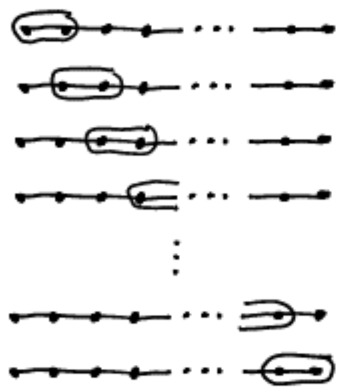


$\Rightarrow$ prob. of 0 that left node is single
" 1 "
" 0 "

Likewise, the first pairing on a chain of n nodes, which occurs across each of the $n-1$ edges w/ equal prob., immediately reduces the problem to the prob. that the node on the left end of a chain of k nodes is single, where k is any integer from 0 to $n-2$ w/ equal prob.

Visually,

equal prob. of $\frac{1}{n-1}$ of each possible first pairing.



$\Rightarrow$ prob. of T_0 that left node is single

" T_1 "

" T_2 "

" T_3 "

$\vdots$

" T_{n-3} "

" T_{n-2} "

Written out mathematically, this is:

$$T_n = \frac{1}{n-1}\left(T_0 + T_1 + T_2 + \cdots + T_{n-3} + T_{n-2}\right)$$
$$= \frac{1}{n-1}\sum_{k=0}^{n-2} T_k.$$

b) Let $U_n = T_n - T_{n-1}$. From eq. (1),

$$(n-1)T_n = \sum_{k=0}^{n-2} T_k \qquad (*)$$

$$(n-2)T_{n-1} = \sum_{k=0}^{n-3} T_k \qquad (**)$$

Subtracting corresponding sides of (**) from (*) gives

$$(n-1)T_n - (n-2)T_{n-1} = T_{n-2} \Rightarrow (n-1)(T_n - T_{n-1}) = -(T_{n-1} - T_{n-2})$$
$$\Rightarrow (n-1)U_n = -U_{n-1} \Rightarrow U_n = -\frac{1}{n-1}U_{n-1}$$

Now, $U_1 = T_1 - T_0 = 1 - 0 = 1$, so $U_k = \frac{(-1)^{k-1}}{(k-1)!}$ (where we've replaced n w/ k)

c) $T_n = T_n - T_0$ (since $T_0 = 0$)

$$= (T_n - T_{n-1}) + (T_{n-1} - T_{n-2}) + (T_{n-2} - T_{n-3}) + \cdots + (T_2 - T_1) + (T_1 - T_0)$$
$$= U_n + U_{n-1} + U_{n-2} + \cdots + U_2 + U_1 = \sum_{k=1}^{n} U_k.$$

d) $\lim_{n\to\infty} T_n = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = e^{-1}$ (by the Maclaurin series for e^x)

e) The problem indicates the prob. that v_k remains single is $T_k T_{n-k-1}$.
Let $k' = n-k+1$. Then as $k, k' \to \infty$, the prob. that v_k remains single is
$\lim_{k,k'\to\infty} T_k T_{k'} = (\lim_{k\to\infty} T_k)(\lim_{k'\to\infty} T_{k'}) = e^{-1}\cdot e^{-1} = e^{-2}$.

$\boxed{3.}$

$\boxed{4.}$

a) Let n denote the sum of the sizes of all 3 countries (where size is left open to interpretation, but could mean total # of people for example.)

If each country is of equal size:

$$U = -\frac{\#\text{ of balanced }\Delta\text{'s} - \#\text{ of unbalanced }\Delta\text{'s}}{\text{total }\#\text{ of }\Delta\text{'s}}$$

$$= -\frac{\left[\binom{n}{3} - \left(\frac{n}{3}\right)^3\right] - \left(\frac{n}{3}\right)^3}{\binom{n}{3}}.$$

$$\lim_{n\to\infty} U = -\lim_{n\to\infty} \frac{\left[\binom{n}{3} - \left(\frac{n}{3}\right)^3\right] - \left(\frac{n}{3}\right)^3}{\binom{n}{3}}$$

$$= -\frac{\left(1/6 - 1/27\right) - 1/27}{1/6} = -\frac{5}{9} \quad (\in [-1,1])$$

b) Let

x: fraction of n in country 1

y: " 2

z: " 3

Then

$$U = -\frac{\left[\binom{n}{3} - n^3 xyz\right] - n^3 xyz}{\binom{n}{3}} = \frac{2n^3 xyz - \binom{n}{3}}{\binom{n}{3}}.$$

Observe: max of $f(x,y,z) = xyz$ subject to the constraints

$$g(x,y,z) = x+y+z = 1 \quad ; \quad x,y,z \geq 0$$

gives max of U. Taking x,y,z to vary continuously,

$$\nabla f = -\lambda \nabla g \Rightarrow \begin{cases} yz = -\lambda \\ xz = -\lambda \\ xy = -\lambda \end{cases} \Rightarrow x = y = z \text{ OR } 2 \text{ of } x,y,z \text{ are } 0.$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$f(1/3, 1/3, 1/3) = \frac{1}{27} \qquad f(0,0,1) = 0$$

So both f & U have their max. when each country is of equal size.

$\boxed{5.}$

a) Differentiating (2) by m using the chain rule, we find

$$\frac{dA(mn)}{dm} = \frac{dA(mn)}{d(mn)} \frac{d(mn)}{dm} = \frac{dA(m)}{dm} \Rightarrow \frac{1}{n}\frac{dA(m)}{dm} = \frac{dA(mn)}{d(mn)}. \quad (*)$$

Similarly, differentiating (2) by n gives

$$\frac{dA(mn)}{dn} = \frac{dA(mn)}{d(mn)} \frac{d(mn)}{dn} = \frac{dA(n)}{dn} \Rightarrow \frac{1}{m}\frac{dA(n)}{dn} = \frac{dA(mn)}{d(mn)}. \quad (**)$$

Combining (*) & (**) gives

$$\frac{1}{n}\frac{dA(m)}{dm} = \frac{1}{m}\frac{dA(n)}{dn} \Rightarrow m\frac{dA(m)}{dm} = n\frac{dA(n)}{dn}.$$

b) Since (3) holds for arbitrary m & n, both sides must be indep. of m & n. So,

$$n\frac{dA(n)}{dn} = K \quad \text{where } K \text{ is a constant.}$$

Thus,

There is no uncertainty for $n = 1$.

$$\int_{A(1)}^{A(n)} dA(n) = \int_1^n \frac{K}{n'}dn' \Rightarrow A(n) - \cancelto{0}{A(1)} = K(\log n - \cancelto{0}{\log 1}) \Rightarrow A(n) = K\log n.$$