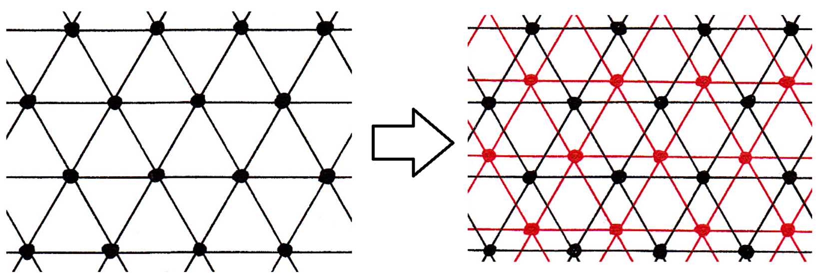
Problem Set 5

Due: Beginning of class (at 2 p.m.) on Monday, Dec. 9.

1) In the notes, we found a bond percolation threshold p_c^b of 1/2 for the square grid graph using its dual graph. The site percolation threshold p_c^s for the triangular grid graph is also 1/2, and here we argue for the plausibility of this using a simple approach which could be regarded as a renormalization argument (perhaps the simplest example of renormalization that can be given).

In site percolation, we make each graph node, or site, "open" with an independent probability p. Two sites are adjacent if they are connected by an edge, and two open sites are in the same cluster if they are adjacent. By Kolmogorov's 0-1 law, the emergence of an infinite cluster occurs at some critical value of p, p_c^s, as we discussed in class. At this transition, there is scale invariance. (Roughly, the system maintains the same statistical appearance as we zoom out.) Let's try to find p_c^s using this property. First, we add a node to the center of every upward pointing triangle and connect each such node to the nearest six new nodes around it as indicated in red below.



This yields a second triangular grid graph identical up to translation to the original. Now suppose that we choose the sites of this second graph to be open if at least 2 of the 3 nearest nodes in the original graph (black) are open. (Unlike the original triangular grid graph, the second triangular grid graph has mild correlations between the "openness" of its adjacent sites. However both open and closed sites exhibit these same correlations, so we will assume they are inconsequential for p_c^s.)

a) Write out an equation for p_c^s expressing the scale invariance at the transition. This should have the form $p_c^s = f(p_c^s)$, where f is the probability that at least 2 of 3 sites in a triangle of the original triangular grid are open (a function in terms of p_c^s).
b) Factor the polynomial found in part (a) and find the roots. What does this suggest for the value of p_c^s?

2) Suppose we have a row of n nodes, where each one (except the last) is linked to the next by an edge. At each time step, we select an edge between unpaired nodes uniformly at random and pair the two nodes it connects. We keep this up until there are no more nodes to pair. In this problem, we will compute the probability that an end node is single and then use this to find the probability that an arbitrary intermediate node is single.

a) Let T_n be the probability that the node on the left end of our chain of n nodes is single. Verify that the first few values of T_n are $T_0 = 0$, $T_1 = 1$, $T_2 = 0$, $T_3 = \frac{1}{2}$, $T_4 = \frac{1}{3}$. Then show that

$$T_k = \frac{1}{n-1} \sum_{k=0}^{n-2} T_k. \tag{1}$$

(Hint: Focus on the fact that in the first time step, the first pairing separates the chain of n nodes into two smaller chains of k and $n-k-2$ nodes, where k may be any integer from 0 to $n-2$ with equal probability.)

b) Now define $U_n = T_n - T_{n-1}$. Use Eq. (1) to show that $U_k = (-1)^{k-1}/(k-1)!$.

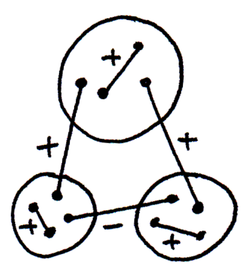
c) Now show that $T_n = \sum_{k=1}^{n} U_k$.

d) Lastly, show that in the large-n limit, T_n equals e^{-1}.

e) Now consider the kth node from the left. Let's call this v_k. If v_k remains single, there are in essence two separate processes going on on either side of v_k, one on a chain of $k-1$ nodes and the other on a chain of $n-k$ nodes. Thus, the probability that v_k remains single is equal to product of the probability that an end node in a chain of length k remains single and the probability that an end node in a chain of length $n-k+1$ remains single. If we let both n and k tend to infinity such that $n-k$ also tends to infinity, what is the probability that v_k is single?

3) A *matching* on a graph $G = (V, E)$ is just a subset of the edges $S \subseteq E$ such that no two edges of S have a node in common and no edge of $E \backslash S$ (E with all the elements of S removed) joins two nodes that are not found in any edge of S. Intuitively, we simply form monogamous pairs on G by some process until we can't form any more such pairs, and S is just the set of those pairs. KORTE78 (see the references of DYER91) gives a proof that the smallest possible S is no smaller than one half the largest possible S. Give an example of a graph for which the smallest possible S is *equal* to one half the largest possible S.

4) Below, the "peacekeeping configuration" (my term) is a *jammed state* in the language of structural balance. (For clarity, only one edge of each type is shown. The sign of that edge is the same for all edges of that type.) This configuration could be interpreted as a system of three countries, where two countries are in a dispute (and hence separated from each other by all negative edges), and a third, mutually friendly country is mediating between the two.



a) Find the energy in the large-n limit for the scenario in which each country is of equal size. (You should obtain a simple fraction independent of n.)

b) Demonstrate that this energy is the highest possible for any peacekeeping configuration. Assume country size is continuous. (Hint: Use Lagrange multipliers.)

5) In this problem, we explore how the first half of Shannon's derivation of the entropy function can be simplified by strengthening property (2) for H to "$A(n)$ is an increasing *and differentiable* function in n." Suppose we have an initial choice between m options and then a second choice between n options (regardless of which of the initial m options we choose). Then by property (3) of H, $A(mn) = A(m) + \sum_{k=1}^{m} \frac{1}{m} A(n)$, or

$$A(mn) = A(m) + A(n). \tag{2}$$

a) Using Eq. (2) and the condition that $A(n)$ is differentiable in n, show that

$$m\frac{dA(m)}{dm} = n\frac{dA(n)}{dn}, \tag{3}$$

for arbitrary m and n.

b) Since Eq. (3) holds for arbitrary m and n, both sides must be independent of m and n. So

$$n\frac{dA(n)}{dn} = K, \tag{4}$$

where K is a constant. Show that this implies $A(n) = K \log n$ (where the logarithm is base e), completing the first half of Shannon's derivation.