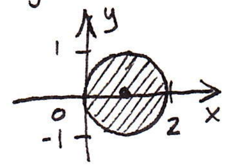
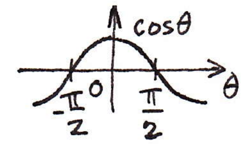
Problem Set 7 - Solutions

15.4.32
$$\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta$$

$$\left\{\begin{array}{l|l} y=\sqrt{2x-x^2} & \Rightarrow x^2+y^2=2x \\ \Rightarrow y^2+(x-1)^2=1 & \Rightarrow r^2=2r\cos\theta \\ & \Rightarrow r=2\cos\theta \end{array}\right.$$





$$= \frac{8}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}\cos\theta(1-\sin^2\theta)\,d\theta \quad \left\{\begin{array}{l} u=\sin\theta \\ du=\cos\theta\,d\theta \end{array}\right.$$
$$= \frac{8}{3}\left[\sin\theta\Big|_0^{\pi/2} - \int_0^1 u^2\,du\right]$$
$$= \frac{8}{3}\left[1-\frac{u^3}{3}\Big|_0^1\right] = \boxed{\frac{16}{9}}$$

15.4.36 (a.)
$$W(R) = \int_0^{2\pi}\int_0^R e^{-r}\,r\,dr\,d\theta$$
$$= 2\pi\int_0^R e^{-r}\,r\,dr \quad \left\{\begin{array}{ll} u=r & du=dr \\ v=-e^{-r} & dv=e^{-r}dr \end{array}\right.$$
$$= 2\pi\left[-re^{-r}\Big|_0^R + \int_0^R e^{-r}\,dr\right]$$
$$= 2\pi\left[-Re^{-R} - e^{-r}\Big|_0^R\right] = \boxed{2\pi\left[1-(1+R)e^{-R}\right]\ ft^3/hr}$$

(b.)
$$\frac{W(R)}{\pi R^2} = \boxed{\frac{2\left[1-(1+R)e^{-R}\right]}{R^2}\ ft/hr}$$

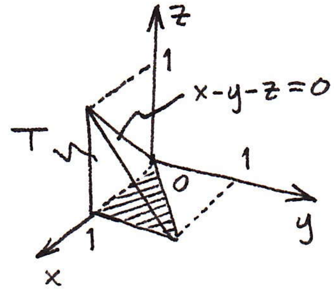
15.4.38
$$\frac{1}{A(D)}\iint_D\sqrt{x^2+y^2}\,dA = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a r^2\,dr\,d\theta = \frac{2}{a^2}\frac{r^3}{3}\Big|_0^a = \boxed{\frac{2}{3}a}$$

15.5.30 (a.) From the material on p. 1034,
$$P(T_1\le 1000, T_2\le 1000) = \int_0^{1000}\int_0^{1000}\frac{1}{1000^2}e^{-(t_1+t_2)/1000}\,dt_1\,dt_2$$
$$= \left(\int_0^{1000}\frac{1}{1000}e^{-t/1000}\,dt\right)^2 = \left(-e^{-t/1000}\Big|_0^{1000}\right)^2 = \boxed{(1-e^{-1})^2} \approx 0.400$$

(b.)
$$P(T_1+T_2\le 1000) = \int_0^{1000}\int_0^{1000-t_1}\frac{1}{1000^2}e^{-(t_1+t_2)/1000}\,dt_2\,dt_1$$
$$= \int_0^{1000}\frac{-1}{1000}e^{-(t_1+t_2)/1000}\Big|_0^{1000-t_1}dt_1 = \int_0^{1000}-\frac{1}{1000}(e^{-1}-e^{-t_1/1000})\,dt_1$$
$$= -e^{-1} - e^{-t_1/1000}\Big|_0^{1000} = \boxed{1-2e^{-1}} \approx 0.264$$

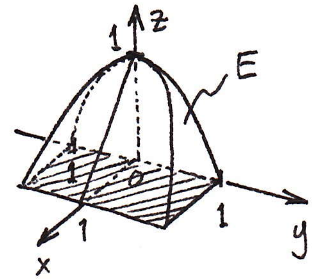
15.7.16 $\iiint_T xyz\,dV = \int_0^1\int_0^x\int_0^{x-y} xyz\,dz\,dy\,dx$

$$= \int_0^1 x\int_0^x y(x-y)^2\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 x\int_0^x (x^2y-2xy^2+y^3)\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 x\left[x\frac{y^2}{2}-2x\frac{y^3}{3}+\frac{y^4}{4}\right]_0^x dx$$
$$= \frac{1}{2}\int_0^1 x\left[\frac{x^4}{2}-\frac{2}{3}x^4+\frac{x^4}{4}\right]dx = \frac{1}{24}\int_0^1 x^5\,dx = \frac{1}{24}\,\frac{x^6}{6}\Big|_0^1 = \boxed{\frac{1}{144}}$$



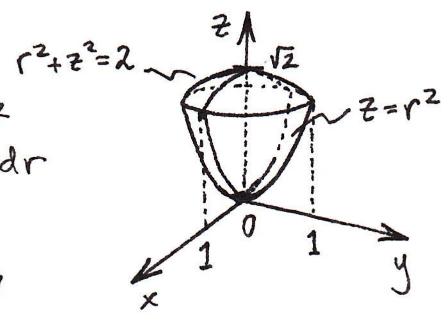
15.7.40 $\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} 4\,dx\,dz\,dy$

$$= 4\int_{-1}^1\left[z-\frac{z^2}{2}\right]_0^{1-y}dy$$
$$= 2\int_{-1}^1(1-y^4)\,dy = 2\left[y-\frac{y^5}{5}\right]_{-1}^1 = \boxed{\frac{16}{5}}$$



15.8.24 $\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$

$$= 2\pi\int_0^1 r\left(\sqrt{2-r^2}-r^2\right)dr \qquad \begin{cases} u = 2-r^2 \\ du = -2r\,dr \end{cases}$$
$$= 2\pi\left[\frac{1}{2}\int_1^2\sqrt{u}\,du - \frac{r^4}{4}\Big|_0^1\right]$$
$$= \pi\,\frac{u^{3/2}}{3/2}\Big|_1^2 - \frac{\pi}{2} = \boxed{\frac{(8\sqrt{2}-7)\pi}{6}} \approx 2.259$$



$$z+z^2=2$$
$$\Rightarrow \left(z+\tfrac{1}{2}\right)^2=\left(\tfrac{3}{2}\right)^2$$
$$\Rightarrow z=1 \Rightarrow r=1$$

15.8.30 $\int_{-3}^3\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

$$= \int_0^{\pi}\int_0^3\int_0^{9-r^2} r^2\,dz\,dr\,d\theta$$
$$= \pi\int_0^3 r^2(9-r^2)\,dr$$
$$= \pi\left[3r^3-\frac{r^5}{5}\right]_0^3 = \boxed{\frac{162\pi}{5}} \approx 101.788$$