Math 255 W13

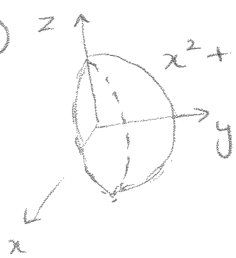
HW 8 Solutions

15.9 (20) E: between spheres $\rho = 1$ & $\rho = 2$ $\Rightarrow$ $1 \le \rho \le 2$

above $z = 0 \iff \varphi = \pi/2$ $\quad 0 \le \varphi \le \pi/2$

not above Quad I $\quad \pi/2 \le \theta \le 2\pi$

$$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2}\int_{\pi/2}^{2\pi}\int_1^2 f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\;d\rho\,d\theta\,d\varphi$$

(24) 

$x^2+y^2+z^2 = 9 \Rightarrow \rho = 3$ $\quad$ (φ: not restricted)

$y \ge 0 \Rightarrow 0 \le \theta \le \pi$

$$\iiint_E y^2\,dV = \int_0^{\pi}\int_0^{\pi}\int_0^{3} \rho^2\sin^2\varphi\,\sin^2\theta\;\rho^2\sin\varphi\;d\rho\,d\theta\,d\varphi$$

$$= \int_0^3 \rho^4\,d\rho \times \int_0^{\pi}\sin^2\theta\,d\theta \times \int_0^{\pi}\sin^3\varphi\,d\varphi$$

$$= \frac{\rho^5}{5}\Big|_0^3 \times \int_0^{\pi}\frac{1-\cos(2\theta)}{2}\,d\theta \times \int_0^{\pi}\sin\varphi(1-\cos^2\varphi)\,d\varphi$$

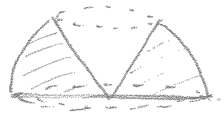
$u = \cos\varphi$

$-du = \sin\varphi\,d\varphi$

$$= \frac{3^5}{5} \times \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{\pi} \times \left[-\cos\varphi + \frac{1}{3}\cos^3\varphi\right]_0^{\pi}$$

$$= \frac{243}{5} \times \frac{\pi}{2} \times \left[1 - \frac{1}{3} - \left(-1+\frac{1}{3}\right)\right] = \frac{162\pi}{5}$$

30



$$z = \sqrt{x^2+y^2} \Rightarrow \varphi = \pi/4$$

$$x^2+y^2+z^2 = 4 \Rightarrow \rho = 2 \qquad (\theta : \text{unrestricted})$$

$$z = 0 \Rightarrow \varphi = \pi/2$$

$$\text{Vol} = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^2 \rho^2 d\rho \times \int_0^{2\pi} d\theta \times \int_{\pi/4}^{\pi/2} \sin\varphi \, d\varphi$$

$$= \frac{\rho^3}{3}\Big|_0^2 \times \theta\Big|_0^{2\pi} \times \Big[-\cos\varphi\Big]_{\pi/4}^{\pi/2}$$

$$= \frac{8}{3}(2\pi)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{8\sqrt{2}\,\pi}{3}$$

40 $-\sqrt{a^2-x^2-y^2} \le z \le \sqrt{a^2-x^2-y^2} \Rightarrow x^2+y^2+z^2 \le a^2$

$\Rightarrow \rho \le |a|$

$$\left.\begin{array}{c} -\sqrt{a^2-y^2} \le x \le \sqrt{a^2-y^2} \\ -a \le y \le a \end{array}\right\} \text{Entire circular disk } x^2+y^2 \le a^2$$

So we are integrating over the entire spherical region.

φ, θ : unrestricted

$$f = x^2 z + y^2 z + z^3 = z(x^2+y^2+z^2) = \rho^3 \cos\varphi$$

$$\text{Integral} = \int_0^{\pi}\int_0^{2\pi}\int_0^{|a|} (\rho^3\cos\varphi)\, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

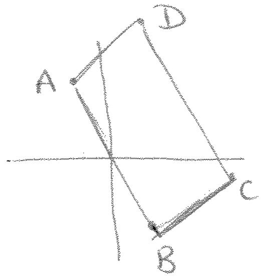
$$= \int_0^{|a|} \rho^5 d\rho \times \int_0^{2\pi} d\theta \times \int_0^{\pi} \underbrace{\sin\varphi}_{=u} \, \underbrace{\cos\varphi \, d\varphi}_{=du}$$

$$= \frac{\rho^6}{6}\Big|_0^{|a|} \times 2\pi \times \frac{\sin^2\varphi}{2}\Big|_0^{\pi}$$

$$= \frac{a^6}{6} \times 2\pi \times 0$$

$$= 0$$

15.10 (16)



$A(-1,3)$ $B(1,-3)$ $C(3,-1)$ $D(1,5)$

Lines AB & CD are $3x+y=0$ & $3x+y=8$

Lines AD & BC are $y-x=4$ & $y-x=-4$

Given $x = \frac{u+v}{4}$ & $y = \frac{v-3u}{4}$

$\Rightarrow 3x+y = v$ & $y-x = -u$

$\therefore$ Boundaries for S in uv plane are

$v=0$, $v=8$, $u=-4$, $u=4$ $\Rightarrow [-4,4]\times[0,8]$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \frac{1}{16} + \frac{3}{16} = \frac{1}{4}$$

$$f = 4x + 8y = u+v+2v-6u = 3v-5u$$

$$\iint_R (4x+8y)\,dA = \int_{-4}^{4}\int_0^8 (3v-5u)\frac{1}{4}\,dv\,du$$

$$= \int_{-4}^{4} \frac{1}{4}\left(\frac{3v^2}{2} - 5uv\right)_0^8 du$$

$$= \int_{-4}^{4} (24-10u)\,du$$

$$= 24u - 5u^2 \Big|_{-4}^{4}$$

$$= 24(8) - 0$$

$$= 192$$

(22) R enclosed by $xy = a, xy = b, xy^{1.4} = c, xy^{1.4} = d$

$0 < a < b, \quad 0 < c < d$

$$W = \text{Area}(R) = \iint_R dA$$

Choose $u = xy \qquad v = xy^{1.4}$

S in uv plane enclosed by $u = a, u = b, v = c, v = d$

$$S = [a, b] \times [c, d]$$

$$\frac{v}{u} = y^{0.4} \Rightarrow y = \left(\frac{v}{u}\right)^{2.5} = u^{-2.5} v^{2.5}$$

$$x = \frac{u}{u^{-2.5} v^{2.5}} = u^{3.5} v^{-2.5}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3.5\, u^{2.5} v^{-2.5} & -2.5\, u^{3.5} v^{-3.5} \\ -2.5\, u^{-3.5} v^{2.5} & 2.5\, u^{-2.5} v^{1.5} \end{vmatrix}$$

$$= \left[(3.5)(2.5) - (2.5)^2\right] v^{-1}$$

$$= \frac{2.5}{v}(3.5 - 2.5) = \frac{2.5}{v} = \frac{5}{2v}$$

$$W = \int_a^b \int_c^d \frac{5}{2v}\, dv\, du$$

$$= \frac{5}{2} \int_a^b du \times \int_c^d \frac{1}{v}\, dv$$

$$= \frac{5}{2}(b-a)\left[\ln v\right]_c^d$$

$$= \frac{5}{2}(b-a)\left(\ln \frac{d}{c}\right)$$

(24) R enclosed by $x-y=0$, $x-y=2$, $x+y=0$, $x+y=3$

Choose $u = x-y$, $v = x+y$

S in uv plane enclosed by $u=0$, $u=2$, $v=0$, $v=3$

$S = [0,2] \times [0,3]$

$$x = \frac{u+v}{2} \qquad y = \frac{v-u}{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$f = (x+y)\, e^{x^2-y^2} = (x+y)\, e^{(x+y)(x-y)} = v e^{uv}$$

$$\iint_R (x+y)\, e^{x^2-y^2}\, dA = \int_0^3 \int_0^2 v e^{uv} \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2} \int_0^3 e^{uv} \Big|_0^2 \, dv$$

$$= \frac{1}{2} \int_0^3 (e^{2v} - 1)\, dv$$

$$= \frac{1}{2} \left[\frac{e^{2v}}{2} - v \right]_0^3$$

$$= \frac{e^6 - 7}{4}$$

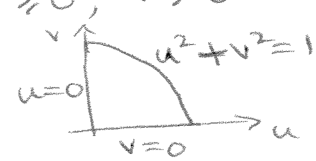
(26) $\iint_R \sin(9x^2 + 4y^2)\, dA$

R. enclosed by $9x^2 + 4y^2 = 1$ $\Rightarrow 9x^2 + 4y^2 \leq 1$

Quad I $\Rightarrow x \geq 0$, $y \geq 0$

Let $u = 3x$ & $v = 2y$ $\Rightarrow$ S : $u^2 + v^2 \leq 1$, $u \geq 0$, $v \geq 0$

$x = u/3 \qquad y = v/2$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3 & 0 \\ 0 & 1/2 \end{vmatrix} = 1/6$$

$$\iint_R \sin(9x^2+4y^2)\, dA = \iint_S \sin(u^2+v^2)\, \frac{1}{6}\, dA_{u,v}$$

$u = r\cos\theta$

$v = r\sin\theta$

$u^2+v^2=1 \Rightarrow r=1$

$u,v \geq 0 \Rightarrow 0 \leq \theta \leq \pi/2$

$$= \int_0^{\pi/2}\int_0^1 \sin(r^2)\, \frac{1}{6}\, r\, dr\, d\theta$$

$$= \frac{1}{6}\int_0^{\pi/2} d\theta \times \int_0^1 \sin(r^2)\, r\, dr$$

$\left(\text{Let } t = r^2 \Rightarrow r\,dr = \frac{dt}{2}\right)$

$$= \frac{1}{6} \times \frac{\pi}{2} \times \left[-\frac{1}{2}\cos(r^2)\right]_0^1$$

$$= \frac{\pi}{24}\left(1-\cos 1\right)$$