Problem Set 9 - Solutions

16.1.24 $\nabla f(x,y,z) = (f_x, f_y, f_z) = \boxed{\left(\ln(y-2z), \frac{x}{y-2z}, \frac{-2x}{y-2z}\right)}$

16.2.6 A natural parametrization of C is: $x(t) = t^3, y(t) = t, -1 \le t \le 1$

$$\int_C e^x dx = \int_{-1}^{1} e^{t^3}(3t^2)dt \quad \text{by Eq. 7, p. 1090}$$
$$= \int_{-1}^{1} e^u du \quad u = t^3, du = 3t^2 dt$$
$$= e^u \Big|_{-1}^{1} = \boxed{e - e^{-1}}$$

16.2.10 A natural parametrization of C is (from Eq. 8, p. 1091):

$\vec{r}(t) = (1-t)(-1,5,0) + t(1,6,4), \quad 0 \le t \le 1$

$\Rightarrow x(t) = 2t-1, \ y(t) = t+5, \ z(t) = 4t, \quad 0 \le t \le 1$

$$\int_C xyz^2 ds = \int_0^1 (2t-1)(t+5)(4t)^2 \sqrt{2^2+1^2+4^2}\, dt = \boxed{236\sqrt{21}/15}$$

16.2.20 $r'(t) = (2t, 3t^2, 2t), \quad 0 \le t \le 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t)) \cdot r'(t)\, dt$$
$$= \int_0^1 \left[(t^2+t^3)2t + (t^3-t^2)3t^2 + (t^2)^2 2t\right] dt$$
$$= \int_0^1 (5t^5 - t^4 + 2t^3)dt = \boxed{17/15}$$

16.3.10 $P = xy\cosh(xy) + \sinh(xy) \Rightarrow P_y = x\cosh(xy) + x^2y\sinh(xy) + x\cosh(xy)$

$Q = x^2\cosh(xy) \Rightarrow Q_x = 2x\cosh(xy) + x^2y\sinh(xy)$

P, Q are defined on all of $\mathbb{R}^2$ & have continuous 1st-order derivatives, so (by Thm 6, p. 1103) F is conservative.

$f = \int f_y dy = \int x^2\cosh(xy)dy = x\sinh(xy) + g(x)$

$f_x = \sinh(xy) + xy\cosh(xy) + g'(x)$

Comparison of f_x w/ $P \Rightarrow g'(x) = 0 \Rightarrow g$ is constant. Take $g = 0$.

So, $\boxed{f = x\sinh(xy)}$

16.3.14 (a.) $f = \int f_y \, dy = \int x^2 e^{xy} \, dy = x e^{xy} + g(x)$

$f_x = e^{xy} + xy e^{xy} + g'(x)$

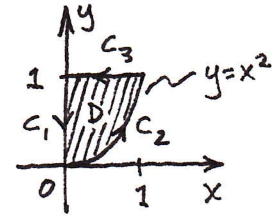
Comparison of f_x w/ $P \Rightarrow g'(x) = 0 \Rightarrow g$ is constant. Take $g = 0$.

So, $\boxed{f = x e^{xy}}$

(b.) $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(\pi/2)) - f(\vec{r}(0))$ (by FTLI)

$= f(0,2) - f(1,0) = 0 - 1 = \boxed{-1}$

16.4.4 (a.) $P = x^2 y^2$, $Q = xy$



Natural parametrizations:

$$\left.\begin{array}{l} C_1 : x(t) = 0, \quad y(t) = 1 - t \\ C_2 : x(t) = t, \quad y(t) = t^2 \\ C_3 : x(t) = 1 - t, \quad y(t) = 1 \end{array}\right\} 0 \le t \le 1$$

$C = C_1 \cup C_2 \cup C_3$

$$\oint_C x^2 y^2 dx + xy\,dy = \int_{C_1} P\,dx + \overbrace{Q\,dy}^{0} + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy$$

$$= \int_0^1 [t^2 (t^2)^2 (1) + t(t^2)(2t)] \, dt + \int_0^1 [(1-t)^2(-1) + (1-t)(1)(0)] \, dt$$

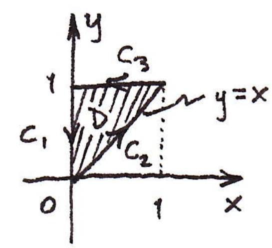
$$= \int_0^1 [t^6 + 2t^4 - (t-1)^2] \, dt = \boxed{22/105}$$

(b.) $\oint_C x^2 y^2 dx + xy\,dy = \iint_D (Q_x - P_y) \, dA$ (by Green's Thm)

$$= \int_0^1 \int_{x^2}^1 (y - 2x^2 y) \, dy \, dx = \int_0^1 (1 - 2x^2) \frac{y^2}{2} \Big|_{x^2}^1 dx$$

$$= \int_0^1 \left(x^6 - \frac{x^4}{2} - x^2 + \frac{1}{2}\right) dx = \boxed{22/105}$$

16.4.14 $P = \sqrt{x^2+1}$, $Q = \tan^{-1} x$



$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y) \, dA$$

$$= \int_0^1 \int_x^1 \left(\frac{1}{1+x^2} - 0 \right) dy \, dx$$

$$= \int_0^1 \frac{1-x}{1+x^2} \, dx$$

$C = C_1 \cup C_2 \cup C_3$

$$= \tan^{-1} x \Big|_0^1 - \frac{1}{2} \int_1^2 \frac{du}{u} \qquad u = 1 + x^2, \; du = 2x\,dx$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln u \Big|_1^2 = \boxed{\frac{\pi}{4} - \frac{1}{2} \ln 2}$$