Math 255 - W13

HW 10 Solutions

16.5 (6) (a)

$$\operatorname{curl} \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & e^{xy}\sin z & y\tan^{-1}\left(\frac{x}{z}\right) \end{vmatrix}$$

$$= \left\langle \tan^{-1}\left(\frac{x}{z}\right) - e^{xy}\cos z,\ -y\frac{1}{1+\left(\frac{x}{z}\right)^2}\frac{1}{z},\ ye^{xy}\sin z \right\rangle$$

$$= \left\langle \tan^{-1}\left(\frac{x}{z}\right) - e^{xy}\cos z,\ \frac{-yz}{x^2+z^2},\ ye^{xy}\sin z \right\rangle$$

(b) $$\operatorname{div} \bar{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}\left(e^{xy}\sin z\right) + \frac{\partial}{\partial z}\left(y\tan^{-1}\left(\frac{x}{z}\right)\right)$$

$$= 0 + xe^{xy}\sin z + y\frac{1}{1+\left(\frac{x}{z}\right)^2}\left(-\frac{x}{z^2}\right)$$

$$= xe^{xy}\sin z - \frac{xy}{x^2+z^2}$$

(18)

$$\operatorname{curl} \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x\sin yz & ze^x\cos yz & ye^x\cos yz \end{vmatrix}$$

$$= \langle e^x\cos yz - yze^x\sin yz - (e^x\cos yz - yze^x\sin yz),$$
$$-(ye^x\cos yz - ye^x\cos yz),$$
$$ze^x\cos yz - ze^x\cos yz \rangle$$

$= \bar{0}$ on $\mathbb{R}^3$: open & simply-connected

$\Rightarrow \bar{F}$ is conservative.

$$f = \int e^x \sin yz\, dx = e^x\sin yz + g(y,z)$$

$$\Rightarrow f_y = ze^x\cos yz + g_y$$

Compare with $Q \Rightarrow g_y = 0 \Rightarrow g(y,z) = h(z)$.

$\Rightarrow f_z = ye^x \cos yz + h'$

Compare with $R \Rightarrow h' = 0 \Rightarrow h = c$. Let $c = 0$

A potential fn is $f = e^x \sin yz$.

15.6

(12) $x^2+y^2+z^2 = 4z$, inside $z = x^2+y^2 \Rightarrow z \geq 0$

$\Leftrightarrow x^2+y^2+(z-2)^2 = 4 \Rightarrow z = 2+\sqrt{4-x^2-y^2} = g(x,y)$

$g_x = \dfrac{-2x}{2\sqrt{4-x^2-y^2}} = \dfrac{-x}{\sqrt{4-x^2-y^2}}$ & $g_y = \dfrac{-y}{\sqrt{4-x^2-y^2}}$

$\sqrt{g_x^2+g_y^2+1} = \sqrt{\dfrac{x^2+y^2+(4-x^2-y^2)}{4-x^2-y^2}} = \dfrac{2}{\sqrt{4-x^2-y^2}}$

$x^2+y^2+z^2 = 4z \cap z = x^2+y^2 \Rightarrow z^2 = 3z \Rightarrow$ ~~[illegible]~~ $z = 0, 3$

On S, $0 \leq z \leq 3 \Rightarrow x^2+y^2 \leq 3$ is D

Surface Area $= \iint_S dS = \iint_D \dfrac{2}{\sqrt{4-x^2-y^2}}\, dA$

$= \int_0^{2\pi}\int_0^{\sqrt{3}} \dfrac{2}{\sqrt{4-r^2}}\, r\,dr\,d\theta$ $\quad u = 4-r^2$

$= 2\pi\left[-2\sqrt{4-r^2}\right]_0^{\sqrt{3}} = 4\pi$

16.6

(34) $\bar{r}(u,v) = \langle u^2+1, v^3+1, u+v\rangle$

$$\left.\begin{aligned} &u^2+1 = 5 \Rightarrow u = -2 \text{ or } 2 \\ &v^3+1 = 2 \Rightarrow v = 1 \\ &u+v = 3 \Rightarrow u = 2 \text{ (not } -2) \end{aligned}\right\} \Rightarrow \bar{r}(2,1) = \langle 5,2,3\rangle$$

$\bar{r}_u = \langle 2u, 0, 1\rangle \Rightarrow \bar{r}_u(2,1) = \langle 4,0,1\rangle$

$\bar{r}_v = \langle 0, 3v^2, 1\rangle \Rightarrow \bar{r}_v(2,1) = \langle 0,3,1\rangle$

$$\bar{n} = \bar{r}_u(2,1) \times \bar{r}_v(2,1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} = \langle -3, -4, 12 \rangle$$

Eq^n of tangent plane is

$$-3(x-5) - 4(y-2) + 12(z-3) = 0$$

$$12z = 3x + 4y + 13$$

(42) $S: \bar{r}(x,y) = \langle x, y, \underbrace{\sqrt{x^2+y^2}}_{g(x,y)} \rangle$

$D: \quad x^2 \le y \le x$

$0 \le x \le 1$

$$g_x = \frac{x}{\sqrt{x^2+y^2}} \qquad g_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$|\bar{r}_x \times \bar{r}_y| = \sqrt{g_x^2 + g_y^2 + 1} = \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1} = \sqrt{2}$$

$$\text{Surface Area} = \iint_S dS = \iint_D |\bar{r}_x \times \bar{r}_y| \, dA$$

$$= \int_0^1 \int_{x^2}^{x} \sqrt{2} \, dy \, dx$$

$$= \sqrt{2} \int_0^1 (x - x^2) \, dx$$

$$= \sqrt{2} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \sqrt{2} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\sqrt{2}}{6}$$

16.7 (16) $S: x^2+y^2+z^2 = 4, \quad x^2+y^2 \le 1 \ \& \ z \ge 0$

$\Rightarrow z = \sqrt{4-x^2-y^2} = g(x,y), \ D: x^2+y^2 \le 1$

$\bar{r}(x,y) = \langle x, y, \sqrt{4-x^2-y^2} \rangle$

$$g_x = \frac{-x}{\sqrt{4-x^2-y^2}} \qquad g_y = \frac{-y}{\sqrt{4-x^2-y^2}}$$

$$|\bar{r}_x \times \bar{r}_y| = \frac{2}{\sqrt{4-x^2-y^2}}$$

$$\iint_S y^2\,dS = \iint_D y^2\,|\bar{r}_x \times \bar{r}_y|\,dA_{xy}$$

$$= \int_0^{2\pi}\int_0^1 r^2\sin^2\theta\,\frac{2}{\sqrt{4-r^2}}\,r\,dr\,d\theta$$

$$\left(\begin{array}{l} u = 4-r^2 \\ -du = 2r\,dr \end{array}\right)$$

$$= \int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta \times \int_4^3 \frac{4-u}{\sqrt{u}}\,(-du)$$

$$= \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi} \times \left[8\sqrt{u} - \frac{2}{3}u^{3/2}\right]_3^4$$

$$= \pi\left(16 - \frac{16}{3} - (8\sqrt{3} - 2\sqrt{3})\right) = \pi\left(\frac{32}{3} - 6\sqrt{3}\right)$$

(24) $S: z = \sqrt{x^2+y^2}$ & $1 \le z \le 3 \Rightarrow 1 \le x^2+y^2 \le 9$

$g(x,y) = \sqrt{x^2+y^2}$ $\qquad \bar{r}(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle$

$\bar{r}_x \times \bar{r}_y = \langle -g_x, -g_y, 1\rangle$ upward since z-comp > 0

$$= \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right\rangle$$

S has downward orientation,

so $d\bar{S} = -(\bar{r}_x \times \bar{r}_y)\,dA_{xy}$

$$\iint_S F\cdot d\bar{S} = -\iint_D \langle -x, -y, (x^2+y^2)^{3/2}\rangle \cdot \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right\rangle dA_{xy}$$

$$= -\iint_D \frac{x^2+y^2+(x^2+y^2)^2}{\sqrt{x^2+y^2}}\,dA$$

$$= -\int_0^{2\pi}\int_1^3 (r+r^3)\,r\,dr\,d\theta$$

$$= -2\pi\left[\frac{r^3}{3} + \frac{r^5}{5}\right]_1^3$$

$$= -2\pi\left[\frac{26}{3} + \frac{242}{5}\right] = -\frac{1712\pi}{15}$$

(28) $\bar{F} = \langle xy, 4x^2, yz \rangle$

$S: \bar{r}(x,y) = \langle x, y, xe^y \rangle \qquad D: [0,1] \times [0,1]$

$\bar{r}_x \times \bar{r}_y = \langle -e^y, -xe^y, 1 \rangle$ upward, as required

$\bar{F} \cdot (\bar{r}_x \times \bar{r}_y) = -xye^y - 4x^3e^y + yz$

On S, $z = xe^y$

$\Rightarrow \bar{F} \cdot (\bar{r}_x \times \bar{r}_y) = -4x^3e^y$

$$\iint_S \bar{F} \cdot d\bar{S} = \int_0^1 \int_0^1 -4x^3e^y \, dy \, dx$$

$$= -4 \int_0^1 x^3 \, dx \times \int_0^1 e^y \, dy$$

$$= -x^4 \Big|_0^1 \times e^y \Big|_0^1$$

$$= 1 - e$$